NEW INTERMEDIATE CO-ORDINATE GEOMETRY

5813.



Rene Descartes (1596—1650), a French mathematician, the "father" of Analytical Geometry, invented the basis of the subject from his dream on the night of 10th November, 1619.

NEW INTERMEDIATE
CO-ORDINATE GEOMETRY

ACCORDING TO THE NEW SYLLABUS

by
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PREFACE TO THE TENTH EDITION

From the very rapid sale of the first nine heavy editions it appears that the book has really been found useful and supplied a long-felt need. On account of the questions set recently on the angle of intersection of two circles, a method *to find the angle of intersection of two circles*, which was given in the ninth edition, has been retained. (See Misc. Exs. 36, 37, Chap. V.) Similarly the chapter on *"Pair of Straight Lines Represented by a Quadratic Equation"* (Chap. IV) (**which should be omitted by the Intermediate students of the Punjab University**) has been retained to cover the syllabi of all the other Indian Universities for the Intermediate Examinations in Co-ordinate Geometry.

In view of the recently raised standard of Examinations the book has been thoroughly revised, recast at some places and brought up to date. Slight modifications have been effected in the wording of some of the examples set in the University Examinations to make them of more permanent value. The sets of examples after the articles have been further enriched with modern questions. Copious hints and complete solutions have been given to typical examples. But in view of the new style of Examination papers requiring ingenuity and skill on the part of the examinees to try examples, most of the hints have been given **at the end of the book** and not side by side with the examples themselves. This arrangement, it is hoped, will minimise the chances of temptation to the students to see the hints without giving a fair trial to the examples, and thus the book in its present form will be found more useful to those for whom it is meant.

D.A.-V. College, Jullundur City,
2nd October, 1957

BANSI LAL

PREFACE TO THE THIRD EDITION

Every one, who writes a book, does so with a definite purpose. My purpose of writing this book is to make the 'Elements of Co-ordinate Geometry' easily comprehensible to the beginners and to make the students well-grounded in the subject. The book has the following

Distinguishing Features.

(1) It *just* covers the University syllabus and is complete in itself, rigidly excluding all matters, however important they might appear from *other* points of view.

(2) Written primarily from the *Examination stand-point*, it attempts to present the subject in an *'easy-to-understand'* and *'easy-to-reproduce'* form. For this reason, the subject-matter illustrated by a *free use* of the diagrams has been put in as simple and lucid a manner as possible.

(3) Wherever possible, the method has been summarised into a *Rule*, and, where necessary, an **Aid to memory** has been provided.

(4) Each article has been followed by *typical* examples. This will facilitate the task of the teacher while allotting work to the class after an

article or two have been done in the class-room. *For the more ambitious and voracious reader*, each chapter ends with a set of *Miscellaneous Examples* (not to form *necessarily* a part of the class-work).

(5) *All* the questions set in the Punjab University from 1924 up to date have been incorporated in the book and last five years' also appended at the end. In addition to these a very large number of examples has been taken from the Examination papers of such other Universities as include Co-ordinate Geometry in their syllabus.

Copious hints and, in most cases, complete solutions of the important examples have been given.

All this is done to give the students an idea of the type of questions usually set in University Examinations and of the type of solutions usually liked by the Examiners.

(6) All articles and examples which are most important from the *Examination stand-point* have been printed in bold type and those next to them in importance in italics. This will help the student in his judicious choice of them when the time at his disposal is limited.

(7) Straight Line and Circle have been very fully treated. Each method of the Circle is *general* which, if mastered, will render the study of the Parabola, the Ellipse and the Hyberbola very easy.

A method in which the *Geometry* of the Circle simplifies matters considerably has been designated as *Shorter method.*

(8) Interesting and instructive alternative proofs have been given here and there for brilliant students to appreciate different methods of approach. 'Which of them is the best and why' has been indicated in '*Comparison of the two methods*'. For the more intelligent students is intended all the matter marked with a double asterisk (**).

(9) A chapter-wise list of all the important results and formulae to be remembered by the student has been prefixed to the book for ready reference and occasional review.

In short, no pains have been spared to make the book useful for all types of students—ordinary, average and brilliant.

In fine, acknowledgment is due, and is hereby made to all the authors consulted in the preparation of the book.

Suggestions for the improvement of the book as well as intimations of errors and serious misprints will be most gratefully received.

D.A.-V. College,
Jullundur City.

BANSI LAL

CONTENTS

CHAP. SEC.		PAGE
	Syllabi of Punjab and Jammu & Kashmir Universities ...	*xi*
	Abbreviations	*xii*
	Important Results and Formulae	*xiii*
I.	**The Point**	
I.	Directed Lines. Rectangular Axes	1
II.	Distance between Two Points	4
III.	Co-ordinates of Points dividing the join of two Given Points in a Given Ratio	6
IV.	Area of a Triangle	14
	Miscellaneous Examples on Chapter I	18
II.	**Locus and its Equation**	
I.	Equation of a Locus	21
II.	Locus of an Equation	23
III.	**The Straight Line**	
I.	Equation of a Straight Line in Different Forms ...	25
II.	Angle between Two Straight Lines	47
III.	Intersection of Straight Lines	57
IV.	Length of the Perpendicular from a Given Point on a Given Line	66
**V.	A Line and a Point	70
VI.	Bisectors of Angles between Two Straight Lines ...	71
	Miscellaneous Examples on Chapter III	75
IV.	**Pair of Straight Lines Represented by a Quadratic Equation (to be omitted by the Intermediate students of the Punjab University)**	
I.	Equation of a Pair of Straight Lines	80
II.	Angle formula for a Pair of Straight Lines through the Origin	84
III.	Angle-Bisectors of a Pair of Straight Lines through the Origin	86
IV.	General Equation of the Second Degree Representing a Pair of Straight Lines	90
	Miscellaneous Examples on Chapter IV	98
V.	**The Circle**	
I.	Equation of a Circle in Different Forms	100
II.	Tangents and Normals	107
III.	A Circle and a Line	113

IV. A Circle and a Point. Pole and Polar ... 121, 126
Use of Parameter 131
V. Geometrical Properties 134
VI. Two or more Circles. Radical Axis of Two Circles 138
Miscellaneous Examples on Chapter V 147

VI. The Parabola

I. Equation of a Parabola 153
II. Tangents and Normals 158
III. A Parabola and a Line 164
IV. A Parabola and a Point. Pole and Polar ... 168, 171
Use of Parameter 173
V. Geometrical Properties 175
Miscellaneous Examples on Chapter VI 178

VII. The Ellipse

I. Equation of an Ellipse 184
II. Tangents and Normals 193
III. An Ellipse and a Line 196
IV. An Ellipse and a Point. Pole and Polar ... 200, 204
Use of Parameter 206
V. Geometrical Properties 210
Miscellaneous Examples on Chapter VII 219

VIII. The Hyperbola (to be omitted by the Intermediate students of the Jammu & Kashmir University)

I. Equation of a Hyperbola 225
II. Tangents and Normals 232
III. A Hyperbola and a Line 232
IV. A Hyperbola and a Point. Pole and Polar ... 233, 236
Use of Parameter 237
V. Geometrical Properties 239
Miscellaneous Examples on Chapter VIII 241

Appendix 244
Punjab University Papers 245
Answers and Hints ... ... *i*

PUNJAB* UNIVERSITY SYLLABUS IN ANALYTICAL CONICS FOR THE INTERMEDIATE EXAMINATION, 1958 (*and after*)

Directed lines. Rectangular axes. Distance between two points. Co-ordinates of a point dividing the join of two given points in a given ratio. Area of a triangle. Equations of a straight line in different forms. Length of the perpendicular from a given point on a given line. Intersection of straight lines. Angle between two straight lines. Bisectors of angles between two straight lines.

Elementary properties of the loci given by equations :

$x^2+y^2=a^2$; $(x-h)^2+(y-k)^2=r^2$; $y^2=4ax$; $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

including the use of parameters ; properties of pole and polar, radical axis of two circles (excluding the properties of systems of coaxal circles). Equations of tangents and normals of the above loci.

JAMMU & KASHMIR† UNIVERSITY SYLLABUS IN ANALYTICAL CONICS FOR THE INTERMEDIATE EXAMINATION, 1958 (*and after*)

Directed lines. Rectangular axes. Distance between two points. Co-ordinates of a point dividing the join of two given points in a given ratio. Area of a triangle. Equations of a straight line in different forms. Length of the perpendicular from a given point on a given line. Intersection of straight lines. Angles between two straight lines. Bisectors of angles between two straight lines.

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including the use of parameters, radical axis of two circles (excluding the properties of systems of coaxal circles). Equations of tangents and normals of the above loci.

Five-twelfths of **Intermediate Mathematics Paper A.**

†*Two-fifths of* **Intermediate Mathematics Paper A.**

‡**Omitted for 1959 (and after).**

ABBREVIATIONS

A. B.	stands	for	*Ajmer Board*
A. U.	,,	,,	*Allahabad University*
Ag.U.	,,	,,	*Agra University*
B.	,,	,,	*B. A. Examination*
B.U.	,,	,,	*Bombay University*
Bar. U.	,,	,,	*Baroda University*
B.H.U.	,,	,,	*Benares Hindu University*
C. U.	,,	,,	*Calcutta University*
D. U.	,,	,,	*Delhi University*
Em.	,,	,,	*Emergency Examination*
J. & K. U.	,,	,,	*Jammu & Kashmir University*
M. B. B.	,,	,,	*Madhya Bharat Board*
M. U.	,,	,,	*Madras University*
N. U.	,,	,,	*Nagpur University*
P. U.	,,	,,	*Punjab University*
Pesh. U.	,,	,,	*Peshawar University*
P (P). U.	,,	,,	*Punjab (Pakistan) University*
R. B.	,,	,,	*Rajputana Board*
S.	,,	,,	*Supplementary Examination*
w.r.t.	,,	,,	*with respect to*

Note. If no year is mentioned, it means that the question **was set** *before* 1930.

IMPORTANT RESULTS AND FORMULAE

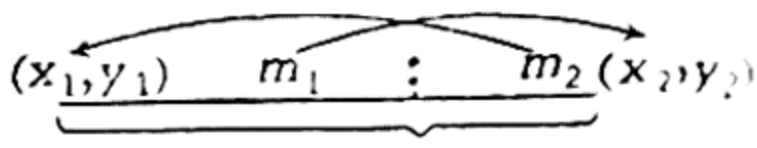
(To Be Remembered)

CHAP. I. THE POINT

1. Distance formula. The distance between the two points (x_1, y_1) and (x_2, y_2) is $\mathbf{d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$. [Art. 5]

2. Section formulae. (*i*) **Internal division.** The co-ordinates of the point which divides the join of the two points (x_1, y_1) and (x_2, y_2) **internally** in the ratio $m_1 : m_2$ are

$$\mathbf{x=\frac{m_1x_2+m_2x_1}{m_1+m_2}},\quad \mathbf{y=\frac{m_1y_2+m_2y_1}{m_1+m_2}}.$$



[Art. 6-I]

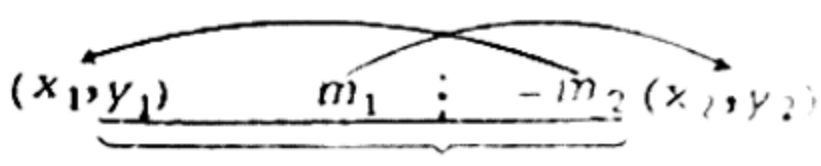
[Remember the **Rule** to write down these from the Fig. (See p. 7.)]

Mid-point formulae. The co-ordinates of the mid-point of the join of (x_1, y_1) and (x_2, y_2) are

$$\mathbf{x=\frac{x_1+x_2}{2}},\quad \mathbf{y=\frac{y_1+y_2}{2}}.$$ [Art. 6-I, Cor.]

(*ii*) **External division.** The co-ordinates of the point which divides the join of the two points (x_1, y_1) and (x_2, y_2) **externally** in the ratio $m_1 : m_2$ are

$$\mathbf{x=\frac{m_1x_2-m_2x_1}{m_1-m_2}},\quad \mathbf{y=\frac{m_1y_2-m_2y_1}{m_1-m_2}}.$$



[Art. 6-II]

[Remember the **Rule** to write down these from the Fig. (See p. 8.)]

3. Centroid formulae. The co-ordinates of the centroid of the triangle, whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), are

$$\mathbf{x=\frac{x_1+x_2+x_3}{3}},\quad \mathbf{y=\frac{y_1+y_2+y_3}{3}}.$$ [Art. 7, (*a*), solved Ex. 1]

4. Incentre formulae. The co-ordinates of the incentre of the triangle, whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, are

$$\mathbf{x=\frac{ax_1+bx_2+cx_3}{a+b+c}},\quad \mathbf{y=\frac{ay_1+by_2+cy_3}{a+b+c}},$$

where $a=BC$, $b=CA$, $c=AB$. [Art. 7, (*b*), solved Ex. 2]

5. Area formula. The area of the triangle, whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), is

$$\triangle=\tfrac{1}{2}[\ (x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)\ ] \quad \ldots \textit{First form}$$

[Remember the **Rule** to write down this expression.]

$$\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}.$$

or $\triangle=\tfrac{1}{2}[\ x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\ ]$. *...Second form* [Art. 9]

Note. Complete area formula. For the **complete** area formula, take the double sign ($\pm$) with the ordinary area formula. [Art. 9, Note 2]

CHAP. III. THE STRAIGHT LINE

1. The equation of the line, **parallel to the x-axis** and at a distance b from it, is $\mathbf{y=b}$. [Art. 15]

The equation of the **x-axis** is $\mathbf{y=0}$. [Art. 15, Cor.]

2. The equation of the line, **parallel to the y-axis** and at a distance a from it, is $\mathbf{x=a}$. [Art. 16]

The equation of the **y-axis** is $\mathbf{x=0}$. [Art. 16, Cor.]

3. The **slope of a line** making an angle θ with the x-axis is $\tan\theta$. [Art. 17]

The slope of a line **equally inclined to the axes is $\mathbf{m=\pm 1}$. [Art. 20, Ex. 3]

4. The equation of the line, **passing through the origin** and having slope m, is $\mathbf{y=mx}$. [Art. 18]

5. **(Slope form or) Tangent form.** The equation of the line, having slope m and cutting off an intercept c on the y-axis, is

$$\mathbf{y=mx+c}.$$ [Art. 20]

6. **Intercept form.** The equation of the line, cutting off intercepts a and b from the axes, is $\dfrac{\mathbf{x}}{\mathbf{a}}+\dfrac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}$. [Art. 21]

Note. The line meets the axes in the points $(\mathbf{a}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{b})$. [Art. 21, Note 1]

7. **Perpendicular form.** The equation of a line, in terms of the length of the perpendicular p from the origin upon it and the angle α which this perpendicular makes with the x-axis, is

$$\mathbf{x\cos\alpha+y\sin\alpha=p}.$$ [Art. 22]

8. The **slope of the line** $Ax+By+C=0$, $=-\dfrac{\mathbf{A}}{\mathbf{B}}$, *i.e.*,

$$\text{slope}=-\frac{\text{coefficient of } \mathbf{x}}{\text{coefficient of } \mathbf{y}}$$ **in the L. H. S. of the equation** of the line (R.H.S. being zero). [Art. 25, Cor.]

9. **Point-slope form.** The equation of the line, passing through (x_1, y_1) and having slope m, is $\mathbf{y-y_1=m(x-x_1)}$. [Art. 28]

The equation of **any line through** $(\mathbf{x_1}, \mathbf{y_1})$ is

$$\mathbf{y-y_1=m(x-x_1)}.$$ [Art. 28, Cor.]

Note. The value of m is found from the second condition satisfied by the line.

10. **Two-point form.** The equation of the line, passing through (x_1, y_1) and (x_2, y_2), is $\mathbf{y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)}$. [Art. 29]

The **slope of the line**, passing through (x_1, y_1) and (x_2, y_2),

$$=\frac{\mathbf{y_2-y_1}}{\mathbf{x_2-x_1}}.$$ [Art. 29, Cor.]

11. **Distance form.** The equation of the line, passing through (x_1, y_1) and making an angle θ with the x-axis, is

$$\frac{\mathbf{x-x_1}}{\mathbf{\cos\ \theta}}=\frac{\mathbf{y-y_1}}{\mathbf{\sin\ \theta}}=\mathbf{r},$$

where r is the distance of the point (x, y) from (x_1, y_1). [Art. 30]

Any point on the line is $(\mathbf{x_1+r\cos\theta,\ y_1+r\sin\theta})$.

[Art. 30, Cor. 2]

12. Angle formula I. Slope form. If θ is the angle between the lines $y=m_1x+c_1$ and $y=m_2x+c_2$, then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}.$$ [Art. 33-I]

Note. Complete angle formula. For the **complete** angle formula (like the complete area formula) take the double sign ($\pm$) with the ordinary angle formula. [Art. 33-I, Note 3]

****Angle formula II. General form.** If θ is the angle between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then

$$\tan\theta=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$ [Art. 33-II]

13. Condition of parallelism I. Slope form. The condition, that the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ may be parallel, is $m_1=m_2$ [*i.e.*, **the slope of one line=the slope of the other**]. [Art. 34-I]

****Condition of parallelism II. General form.** The condition, that the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ may be parallel, is

$$\frac{a_1}{a_2}=\frac{b_1}{b_2},$$

i.e., **ratio of the coefficients of x=ratio of the coefficients of y.**

[Art. 34-II]

14. The equation of **any line parallel to the line** $Ax+By+C=0$ is

$$Ax+By+k=0.$$

[Rule to write down the equation of any line parallel to a given line (equation in the general form) :

In the equation of the given line, change **only** *the constant term to a new constant k.*] [Art. 35]

Note. The value of k is found from the second condition satisfied by the line.

15. Condition of perpendicularity I. Slope form. The condition, that the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ may be perpendicular, is

$$m_1m_2=-1 \text{ or } m_2=-\frac{1}{m_1}$$

[*i.e.*, **the product of their slopes=−1, or the slope of one line =negative reciprocal of the slope of the other**]. [Art. 36-I]

****Condition of perpendicularity II. General form.** The condition, that the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ may be perpendicular, is $a_1a_2+b_1b_2=0$, *i.e.*,

product of the coefficients of x+product of the coefficients of y=0.

[Art. 36-II]

16. The equation of **any line perpendicular to the line** $Ax+By+C=0$ is $Bx-Ay+k=0$.

[Rule to write down the equation of any line perpendicular to a given line (equation in the general form) :

(*i*) *In the equation of the given line, interchange the coefficients of x and y,* (*ii*) *change the sign of one of them, and* (*iii*) *change the constant term to a new constant k.*] [Art. 37]

Note. The value of k is found from the second condition satisfied by the line.

17. Test for concurrency of three lines in special cases. If the equations of three lines are

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0,\ a_3x+b_3y+c_3=0,$$

and $$(a_1x+b_1y+c_1)+(a_2x+b_2y+c_2)+(a_3x+b_3y+c_3)=0$$

identically, the three lines meet in a point. In words :

If adding the L.H.S.'s of the equations of three lines (R.H.S.'s being zero) we get zero, the three lines meet in a point. [Art. 42, Cor.]

18. The equation of **any line through the point of intersection of the two lines** $a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0$ is

$$\mathbf{a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0,}$$

i.e., **one line+k(other line)=0**, where "one line" stands for the "L.H.S. of the equation of one line (R.H.S. being zero)", and so for the "other line." [Art. 43]

Note. The value of k is found from the second condition satisfied by the line.

19. Perpendicular distance formula. The perpendicular distance of the point (x_1, y_1)

(*i*) from the line $x\cos\alpha+y\sin\alpha=p$ is

$$\mathbf{d=x_1\cos\alpha+y_1\sin\alpha-p,}$$ [Art. 44-I]

and (*ii*) from the line $ax+by+c=0$ is

$$\mathbf{d=\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.}$$ [Art. 44-II]

[Rule to find the perpendicular distance of a given point from a given line :

(*i*) *In the L.H.S. of the equation of the line (R.H.S. being zero), substitute the co-ordinates of the point, and* (*ii*) *divide the result by* $\sqrt{(coeff.\ of\ x)^2+(coeff.\ of\ y)^2}$. *The result gives the perpendicular distance.*]

Note. Complete perpendicular distance formula. For the complete perpendicular distance formula (like the complete area formula or the complete angle formula) take the double sign ($\pm$) with the ordinary perpendicular distance formula. [Art. 44-II, Note]

CHAP. IV. PAIR OF STRAIGHT LINES

[☞ *To be omitted by the Intermediate students of the Punjab University.*]

1. If the separate equations of two lines are $ax+by+c=0$, and $a'x+b'y+c'=0$, then their combined equation is

$$\mathbf{(ax+by+c)(a'x+b'y+c')=0.}$$ [Art. 47, Cor.]

2. Two important results. If $y=m_1x$, and $y=m_2x$ are the separate equations of the lines $ax^2+2hxy+by^2=0$, then

$$\mathbf{m_1+m_2=-\frac{2h}{b},\ and\ m_1m_2=\frac{a}{b}.}$$ [Art. 49, Cor. 1]

3. Angle formula for a pair of lines through the origin. If θ is the angle between the lines $ax^2+2hxy+by^2=0$, then

$$\mathbf{\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}.}$$ [Art. 50]

4. Condition of perpendicularity. The condition, that the lines $ax^2+2hxy+by^2=0$ may be perpendicular, is

$$\mathbf{a+b=0},$$

i.e., **coefficient of x^2 + coefficient of y^2 = 0.** [Art. 50, Cor. 1]

****5.** The lines $ax^2+2hxy+by^2=0$ are real and different, real and coincident, or imaginary according as h^2-ab is positive, zero, or negative. [Art. 50, Cor. 2]

6. The **equation of the bisectors of the angles between the lines** $ax^2+2hxy+by^2=0$, is

$$\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}}=\frac{\mathbf{xy}}{\mathbf{h}}.$$ [Art. 51]

7. The condition, that the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ may represent two straight lines, is

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0.}$$ [Art. 52]

****8.** If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, the equation of the lines through the origin parallel to them is $\mathbf{ax^2+2hxy+by^2=0}$, *i.e.*, **second degree terms = 0.** [Art. 52, Cor.]

****9.** The condition, that the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ may make equal angles with the x-axis, is $\mathbf{m_1=-m_2}$,
i.e., **the slope of one line = negative slope of the other.** [Art. 53, Ex. 5]

CHAP. V. THE CIRCLE

1. Standard form. The equation of the circle, whose centre is the origin and radius $=a$, is $\mathbf{x^2+y^2=a^2}$. [Art. 55]

2. Central form. The equation of the circle, whose centre is (h, k) and radius $=a$, is $\mathbf{(x-h)^2+(y-k)^2=a^2}$. [Art. 56]

3. General form. The general equation of a circle is

$$\mathbf{x^2+y^2+2gx+2fy+c=0}\ ;$$

its **centre** is $(-\mathbf{g}, -\mathbf{f})$, and **radius** $=\sqrt{\mathbf{g^2+f^2-c}}$. [Art. 57]

4. Diameter form. The equation of the circle, on the join of (x_1, y_1) and (x_2, y_2) as diameter, is

$$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.}$$ [Art. 60]

5. The **equation of the tangent** at the point (x_1, y_1) of the circle

(*i*) $x^2+y^2=a^2$ is $\mathbf{xx_1+yy_1=a^2}$. [Art. 62]

(*ii*) $x^2+y^2+2gx+2fy+c=0$ is

$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0:}$$ [Art. 64]

[Rule to write down the equation of the tangent at the point (x_1, y_1) of a circle (or conic in whose equation there is no* term involving the product xy) :

In the equation of the curve, change $\mathbf{x^2}$ *to* $\mathbf{xx_1}$, $\mathbf{y^2}$ *to* $\mathbf{yy_1}$, $\mathbf{x}$ *to* $\frac{1}{2}(\mathbf{x+x_1})$, $\mathbf{y}$ *to* $\frac{1}{2}(\mathbf{y+y_1})$.

6. Intercept formula for the circle.

Length of the (chord or) intercept $=2\sqrt{(\text{radius})^2-(\text{central }\perp)^2}$, where central $\perp$ means the $\perp$ from the centre on the line. [Art. 70]

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*If, however, there is a term involving the product xy, change xy to $\frac{1}{2}(xy_1+x_1y)$.

7. Tangent property. If a line touches a circle, the length of the perpendicular from the centre on the line =the radius. [Art. 71]

[☞ *The equation of the circle, when not stated, is taken for granted as* $x^2+y^2=a^2$.]

8. Condition of tangency. The condition, that the line $y=mx+c$ may touch the circle, is $\mathbf{c=\pm a\sqrt{1+m^2}}$. [Art. 71]

If m is the slope of a tangent to the circle, the equation of the tangent is $\mathbf{y=mx\pm a\sqrt{1+m^2}}$. [Art. 71, Cor. 1]

The **equation of any tangent** to the circle, in the m-form, is

$$\mathbf{y=mx+a\sqrt{1+m^2}}.$$ [Art. 71, Cor. 2]

****Condition of tangency.** The condition, that the line $lx+my+n=0$ may touch the circle, is

$$\mathbf{a^2l^2+a^2m^2=n^2}.$$ [Art. 71, Ex. 9]

9. Test for a point to lie outside, on, or inside a circle. The quantity $x_1^2+y_1^2-a^2$ is positive, zero, or negative according as the point (x_1, y_1) lies outside, on, or inside the circle. [Art. 72]

10. The **equation of the chord of contact** of tangents from (x_1, y_1) to the circle is $\mathbf{xx_1+yy_1=a^2}$. [Art. 77]

11. The **equation of the polar** of (x_1, y_1) w.r.t. the circle is

$$\mathbf{xx_1+yy_1=a^2}.$$ [Art. 79]

12. Any point on the circle is (a cos θ, a sin θ). [Art. 82, Cor.]

13. The **equation of the diameter** of the circle, bisecting parallel chords of slope m, is $\mathbf{y=-\frac{1}{m}x}$. [Art. 84]

****14. Equation of a chord in terms of its mid-point.** The equation of the chord of the circle, whose mid-point is (x_1, y_1), is $\mathbf{T=S_1}$, where $S=0$ is the equation of the circle, S_1 is the result of substituting the co-ordinates of the point (x_1, y_1) in S, and $T=0$ is the equation of the tangent at (x_1, y_1). [Art. 84, Ex. 1]

15. The **length of the tangent** from (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is $\mathbf{\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}}$.

[**Rule to find the length of the tangent from a point to a circle :**

In the L.H.S. of the equation of the circle (coefficients of x^2, y^2 *being each=1, R.H.S. being zero), substitute the co-ordinates of the point, and take the square root of the result. The result gives the length of the tangent.*] [Art. 85]

16. The **equation of the radical axis of the two circles**

$$x^2+y^2+2g_1x+2f_1y+c_1=0,$$
$$x^2+y^2+2g_2x+2f_2y+c_2=0,$$

is $\mathbf{2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0}$.

[**Rule to write down the equation of the radical axis of two circles :**

Write the equation of each circle so that the coefficients of x^2, y^2 *are each=1 on the L.H.S., R.H.S. being zero, and subtract one equation from the other. The resulting equation is the equation of the radical axis.*] [Art. 87]

17. Condition of orthogonality. The condition, that the two circles $x^2+y^2+2g_1x+2f_1y+c_1=0$, $x^2+y^2+2g_2x+2f_2y+c_2=0$, may cut orthogonally, is $\mathbf{2g_1g_2+2f_1f_2=c_1+c_2}$. [Art. 91]

18. The equation of **any circle through the points of intersection of the two circles**

$$x^2+y^2+2g_1x+2f_1y+c_1=0,\quad x^2+y^2+2g_2x+2f_2y+c_2=0, \text{ is}$$
$$\mathbf{x^2+y^2+2g_1x+2f_1y+c_1+k(x^2+y^2+2g_2x+2f_2y+c_2)=0,}$$

i.e., **one circle+k(other circle)=0**, where "one circle" stands for the "L.H.S. of the equation of one circle (R.H.S. being zero)", and so for the "other circle." [Art. 92]

Note. The value of k is found from the second condition satisfied by the circle.

19. The equation of **any circle through the points of intersection of the circle** $x^2+y^2+2gx+2fy+c=0$ **and the line** $Ax+By+C=0$, **is** $\mathbf{x^2+y^2+2gx+2fy+c+k(Ax+By+C)=0,}$

i.e., **circle+k(line)=0**, where "circle" stands for the "L.H.S. of the equation of the circle (R.H.S. being zero)", and so for the "line." [Art. 93]

Note. The value of k is found from the second condition satisfied by the circle.

CHAP. VI. THE PARABOLA

1. Definition property. If P is any point on a parabola whose focus is S, and PM ⊥ on the directrix, then

$$\mathbf{SP=PM.}$$ [Art. 94, (*c*)]

2. Standard form. The equation of the parabola is $\mathbf{y^2=4ax}$.

$$\mathbf{AS=AZ=a.}$$ [Art. 95-I]

[☞ *The equation of the parabola, when not stated, is taken for granted as* $y^2=4ax$.]

The **focal distance** of the point $P(x_1, y_1)$ on the parabola is

$$\mathbf{SP=a+x_1.}$$ [Art. 95-I, Cor.]

The focus of the parabola is **(a, 0)**, the equation of the directrix is $\mathbf{x=-a}$, the equation of the axis (x-axis) is $\mathbf{y=0}$, and the vertex (origin) is **(0, 0)**. [Art. 95-I]

3. The **latus rectum** of each of the parabolas $y^2=4ax$, $y^2=-4ax$; $x^2=4ay$, $x^2=-4ay$ is **4a**. [Art. 98]

4. The **equation of the tangent** at the point (x_1, y_1) of the parabola is

$$\mathbf{yy_1=2a(x+x_1).}$$ [Art. 99]

5. The **equation of the normal** at the point (x_1, y_1) of the parabola is

$$\mathbf{y-y_1=-\frac{y_1}{2a}(x-x_1).}$$ [Art. 100]

6. The **equation of any normal** to the parabola, in the m-form, is

$$\mathbf{y=mx-2am-am^3,}$$

and the foot of the normal is $\mathbf{(am^2, -2am)}$. [Art. 101 and Cor. 1]

****7.** In the cubic $ax^3+bx^2+cx+d=0$,

sum of the roots, *i.e.*, $\mathbf{x_1+x_2+x_3=-\frac{b}{a}=-\frac{\text{coeff. of } x^2}{\text{coeff. of } x^3}}$,

sum of the products of the roots taken two at a time,

i.e., $$x_1x_2+x_2x_3+x_3x_1=\frac{c}{a}=\frac{\text{coeff. of } x}{\text{coeff. of } x^3},$$

product of the roots, *i.e.*, $x_1x_2x_3=-\frac{\mathbf{d}}{\mathbf{a}}=-\frac{\textbf{constant term}}{\textbf{coeff. of } \mathbf{x^3}}$. [Art. 102]

8. Condition of tangency. The condition, that the line $y=mx+c$ may touch the parabola, is $\mathbf{c=\frac{a}{m}}$. [Art. 106]

If m is the slope of a tangent to the parabola, the equation of the tangent is $$\mathbf{y=mx+\frac{a}{m}},$$

and the point of contact is $\left(\frac{\mathbf{a}}{\mathbf{m^2}}, \frac{\mathbf{2a}}{\mathbf{m}}\right)$. [Art. 106, Cors. 1, 2]

The **equation of any tangent** to the parabola, in the m-form, is $$\mathbf{y=mx+\frac{a}{m}}.$$ [Art. 106, Cor. 3]

****Condition of tangency.** The condition, that the line $lx+my+n=0$ may touch the parabola, is $\mathbf{am^2=nl}$. [Art. 106, Ex. 4]

9. Test for a point to lie outside, on, or inside a parabola. The quantity $y_1^2-4ax_1$ is positive, zero, or negative according as the point (x_1, y_1) lies outside, on, or inside the parabola. [Art. 107]

10. The equation of the chord of contact of tangents from (x_1, y_1) to the parabola is $\mathbf{yy_1=2a(x+x_1)}$. [Art. 110]

11. The equation of the polar of (x_1, y_1) w.r.t. the parabola is $$\mathbf{yy_1=2a(x+x_1)}.$$ [Art. 111]

12. Any point on the parabola is $(\mathbf{at^2, 2at})$. [Art. 114, Cor.]

**If t_1, t_2 are the parameters of the ends of a focal chord of the parabola, then $\mathbf{t_1t_2=-1}$. [Misc. Ex. 31, Chap. VI]

**The ends of a focal chord of the parabola are

$(\mathbf{at^2, 2at})$ and $\left(\text{changing } t \text{ to } -\frac{1}{t}\right)$

$\left(\frac{\mathbf{a}}{\mathbf{t^2}}, -\frac{\mathbf{2a}}{\mathbf{t}}\right)$. [Misc. Ex. 31, Cor., Chap. VI]

13. The equation of the diameter of the parabola, bisecting parallel chords of slope m, is $\mathbf{y=\frac{2a}{m}}$. [Art. 117, (*d*)]

****14. Equation of a chord in terms of its mid-point.** The equation of the chord of the parabola, whose mid-point is (x_1, y_1), is $\mathbf{T=S_1}$, where $S=0$ is the equation of the parabola, S_1 is the result of substituting the co-ordinates of the point (x_1, y_1) in S, and $T=0$ is the equation of the tangent at (x_1, y_1). [Art. 117, (*d*), Ex.]

CHAP. VII. THE ELLIPSE

[☞ *To be omitted by the Intermediate students of the Jammu & Kashmir University from 1959 (and after).*]

1. Definition property. If P is any point on an ellipse whose focus is S and eccentricity e, and PM $\perp$ on the directrix, then

$$\mathbf{SP = e\ PM.} \quad (e<1) \qquad \text{[Art. 118]}$$

2. Standard form. The equation of the ellipse is

$$\frac{\mathbf{x^2}}{\mathbf{a^2}}+\frac{\mathbf{y^2}}{\mathbf{b^2}}=\mathbf{1}.$$

$$\mathbf{CS=ae,\ CZ=\frac{a}{e},\ b^2=a^2\ (1-e^2).} \qquad \text{[Art. 119,(}a\text{), Cor.]}$$

The foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a^2>b^2)$ are $(\pm\mathbf{ae, 0})$, and the equations of the directrices are $\mathbf{x=\pm\frac{a}{e}}$. [Art. 120, Cor.]

[☞ *The equation of the ellipse, when not stated, is taken for granted as* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.]

3. The **latus rectum** of each of the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \ \frac{x^2}{b^2}+\frac{y^2}{a^2}=1\ (a^2>b^2) \text{ is } \mathbf{2\,\frac{b^2}{a}}. \qquad \text{[Art. 121]}$$

4. The **equation of the tangent** at the point (x_1, y_1) of the ellipse is

$$\frac{\mathbf{xx_1}}{\mathbf{a^2}}+\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1}. \qquad \text{[Art. 122]}$$

5. The **equation of the normal** at the point (x_1, y_1) of the ellipse is

$$\frac{\mathbf{a^2x}}{\mathbf{x_1}}-\frac{\mathbf{b^2y}}{\mathbf{y_1}}=\mathbf{a^2-b^2}. \qquad \text{[Art. 123]}$$

6. Condition of tangency. The condition, that the line $y=mx+c$ may touch the ellipse, is $\mathbf{c=\pm\sqrt{a^2m^2+b^2}}$. [Art. 126]

If m is the slope of a tangent to the ellipse, the equation of the tangent is $\mathbf{y=mx\pm\sqrt{a^2m^2+b^2}}$. [Art. 126, Cor. 1]

The **equation of any tangent** to the ellipse, in the m-form, is

$$\mathbf{y=mx+\sqrt{a^2m^2+b^2}}. \qquad \text{[Art. 126, Cor. 2]}$$

****Condition of tangency.** The condition, that the line $lx+my+n=0$ may touch the ellipse, is $\mathbf{a^2l^2+b^2m^2=n^2}$. [Art. 126, Ex. 6]

7. Test for a point to lie outside, on, or inside an ellipse. The quantity $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ is positive, zero, or negative according as the point (x_1, y_1) lies outside, on, or inside the ellipse. [Art. 127]

8. The **equation of the chord of contact** of tangents from (x_1, y_1) to the ellipse is $\frac{\mathbf{xx_1}}{\mathbf{a^2}}+\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1}$. [Art. 130]

9. The **equation of the polar** of (x_1, y_1) w. r. t. the ellipse is

$$\frac{\mathbf{xx_1}}{\mathbf{a^2}}+\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1}. \qquad \text{[Art. 131]}$$

10. Any point on the ellipse is (a cos θ, b sin θ). [Art. 135, Cor.]

11. The **equation of the normal** at the point (a cos θ, b sin θ) of the ellipse is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. [Art. 136, Ex. 3]

12. The **focal distances** of the point $P(x_1, y_1)$ on the ellipse are $SP=a+ex_1$, $S'P=a-ex_1$. [Art. 137, (a), Cor.]

13. If the normal at the point $P(x_1, y_1)$ of the ellipse meets the major axis in G, then $CG=e^2x_1$. [Art. 137, (d)]

14. The **equation of the diameter** of the ellipse, bisecting parallel chords of slope m, is $y=-\frac{b^2}{a^2m}x$. [Art. 137, (g)]

****15. Equation of a chord in terms of its mid-point.** The equation of the chord of the ellipse, whose mid-point is (x_1, y_1), is $T=S_1$, where $S=0$ is the equation of the ellipse, S_1 is the result of substituting the co-ordinates of the point (x_1, y_1) in S, and $T=0$ is the equation of the tangent at (x_1, y_1).

****16.** The condition, that $y=mx$, $y=m'x$ may be conjugate diameters of the ellipse, is $mm'=-\frac{b^2}{a^2}$. [Art. 137, (g), Ex. 5, Cor. 1]

****The equation of the diameter of the ellipse conjugate to the diameter $y=mx$ is $y=-\frac{b^2}{a^2m}x$. [Art. 137, (g), Ex. 5, Cor. 2]

CHAP. VIII. THE HYPERBOLA

[☞ *To be omitted by the Intermediate students of the Jammu & Kashmir University.*]

1. Definition property. If P is any point on a hyperbola whose focus is S and eccentricity e, and PM $\perp$ on the directrix, then $SP=e\,PM$. $(e>1)$ [Art. 138]

2. Standard form. The equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$ [Art. 139]

$$CS=ae,\ CZ=\frac{a}{e},\quad b^2=a^2(e^2-1).$$ [Art. 139, Cor.]

[**Aid to memory for standard results in the hyperbola** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

In the corresponding results for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, change b^2 to $-b^2$.]

The foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $(\pm ae, 0)$, and the equations of the directrices are $x=\pm\frac{a}{e}$. [Art. 140, Cor.]

[☞ *The equation of the hyperbola, when not stated, is taken for granted as* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.]

3. The **latus rectum** of each of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is $2\frac{\mathbf{b^2}}{\mathbf{a}}$. [Art. 141]

4. The **equation of the tangent** at the point (x_1, y_1) of the hyperbola is $\frac{\mathbf{xx_1}}{\mathbf{a^2}}-\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1}$. [Art. 142, Ex.]

5. The **equation of the normal** at the point (x_1, y_1) of the hyperbola is $\frac{\mathbf{a^2x}}{\mathbf{x_1}}+\frac{\mathbf{b^2y}}{\mathbf{y_1}}=\mathbf{a^2+b^2}$. [Art. 142, Ex.]

6. Condition of tangency. The condition, that the line $y=mx+c$ may touch the hyperbola, is $\mathbf{c=\pm\sqrt{a^2m^2-b^2}}$. [Art. 142, Ex. 5]

If m is the slope of a tangent to the hyperbola, the equation of the tangent is $\mathbf{y=mx\pm\sqrt{a^2m^2-b^2}}$. [Art. 142, Ex. 5, Cor. 1]

The equation of any tangent to the hyperbola, in the m-form, is

$$\mathbf{y=mx+\sqrt{a^2m^2-b^2}}.$$ [Art. 142, Ex. 5, Cor. 2]

****Condition of tangency.** The condition, that the line $lx+my+n=0$ may touch the hyperbola, is $\mathbf{a^2l^2-b^2m^2=n^2}$. [Art. 142, Ex. 9]

7. Test for a point to lie outside, on, or inside a hyperbola. The quantity $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is negative, zero, or positive according as the point (x_1, y_1) lies outside, on, or inside the hyperbola. [Art. 143]

8. The **equation of the chord of contact** of tangents from (x_1, y_1) to the hyperbola is $\frac{\mathbf{xx_1}}{\mathbf{a^2}}-\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1}$. [Art. 143, Ex. 6]

9. The **equation of the polar** of (x_1, y_1) w. r. t. the hyperbola is $\frac{\mathbf{xx_1}}{\mathbf{a^2}}-\frac{\mathbf{yy_1}}{\mathbf{b^2}}=\mathbf{1}$. [Art. 143, Ex. 7, (*b*)]

10. Any point on the hyperbola is (a sec θ, b tan θ). [Art. 144, Cor.]

****11.** The **equation of the normal** at the point (a sec θ, b tan θ) of the hyperbola is $\frac{\mathbf{ax}}{\mathbf{\sec\theta}}+\frac{\mathbf{by}}{\mathbf{\tan\theta}}=\mathbf{a^2+b^2}$. [Art. 144, Ex. 3]

****12.** The **focal distances** of the point $P(x_1, y_1)$ on the hyperbola are $\mathbf{SP=ex_1-a}$, $\mathbf{S'P=ex_1+a}$. [Art. 145, Cor.]

13. If the normal at the point $P(x_1, y_1)$ of the hyperbola meets the transverse axis in G, then $\mathbf{CG=e^2x_1}$. [Art. 145, Ex. 3]

****14.** The **equation of the diameter** of the hyperbola, bisecting parallel chords of slope m, is $\mathbf{y=\frac{b^2}{a^2m}x}$. [Art. 145, Ex. 7]

15. **Equation of a chord in terms of its mid-point. The equation of the chord of the hyperbola, whose mid-point is (x_1, y_1), is $\mathbf{T=S_1}$, where $S=0$ is the equation of the hyperbola, S_1 is the result of substituting the co-ordinates of the point (x_1, y_1) in S, and $T=0$ is the equation of the tangent at (x_1, y_1). [Art. 145, Ex. 9]

**16. The condition, that $y=mx$, $y=m'x$ may be conjugate diameters of the hyperbola, is $\mathbf{mm'=\frac{b^2}{a^2}}$. [Art. 145, Ex. 12, Cor. 1]

**The equation of the diameter of the hyperbola conjugate to the diameter $y=mx$ is $\mathbf{y=\frac{b^2}{a^2m}x}$. [Art. 145, Ex. 12, Cor. 2]

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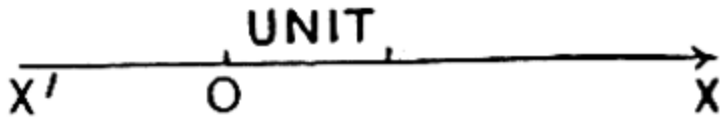
CHAPTER I
THE POINT

SECTION I
DIRECTED LINES. RECTANGULAR AXES

1. Directed line. Def. A **directed line** is a straight line upon which (*i*) a fixed point called the *origin*, (*ii*) a *unit of length*, and (*iii*) a *positive direction* have been assumed.

Notation. The origin is denoted by O (the initial letter of the word *origin*) and the positive direction is indicated by an arrow-head placed upon the directed line.

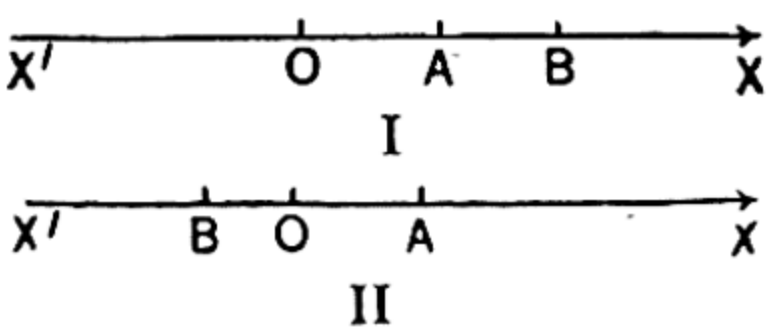
Thus the indefinite straight line X′OX is a **directed line** upon which (*i*) O is the *origin*, (*ii*) the *unit of length* as shown, and (*iii*) the direction OX (*i.e.*, from O to the right) is *positive* [and the direction OX′ (*i.e.*, from O to the left) is negative].



[☞ *All Articles, Examples or Notes marked with a double asterisk* (**) *may be omitted on a first reading of the subject.*]

****Note.** (*A length*) *AB on a directed line is positive if the direction from A to B is the same as the positive direction on the line, and AB is negative if the direction from A to B is the same as the negative direction on the line.*

Thus in Fig. I, AB is positive [∵ the direction from A to B, *i.e.*, to the *right*, is *positive*], and in Fig. II, AB is negative [∵ the direction from A to B, *i.e.*, to the *left*, is *negative*].

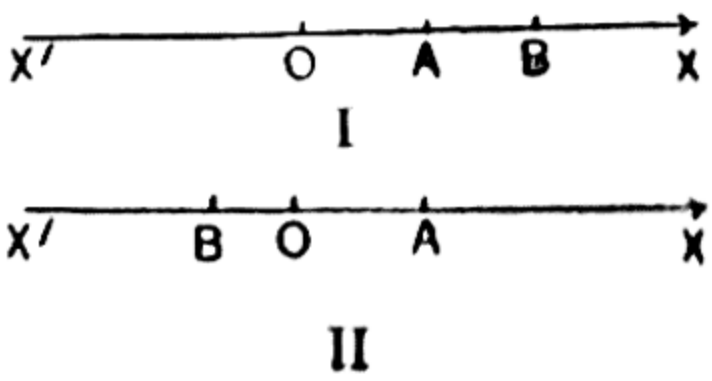


****2. Two fundamental relations.** *For all positions of two points A and B on a directed line,*

(1) $AB = -BA$,
(2) $AB = OB - OA$, *where O is the origin.*

****(1) Proof.** (*i*) *As regards sign,* AB, the length measured from A to B is opposite in sign (direction) to BA, the length measured from B to A.

(*ii*) *As regards magnitude,* AB and BA are equal. [∵ they represent the length of the same st. line]



$\therefore$ from (*i*) and (*ii*), AB$=-$BA*

(2) **Proof. OA + AB† means that the pt. moves from O to A and then from A to B,

i.e., the pt. moves from O to B

$\therefore$ OA$+$AB$=$OB,

or, transposing,

AB$=$OB$-$OA.

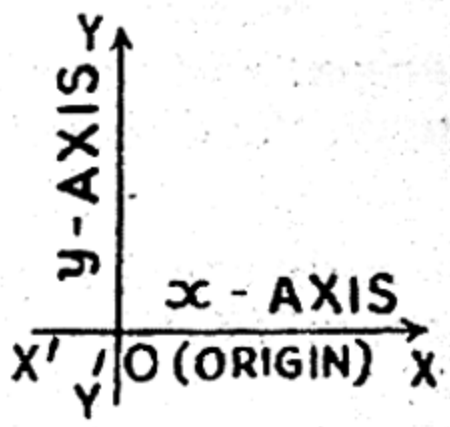
[**Note. How to draw the two Figs.** In Fig. I, the points A, B lie on the same (right) side of O, and in Fig. II, they lie on opposite sides of O (A on the right and B on the left side of O).]

Cor. *If OA$=a$, OB$=b$, then for all positions of two points A and B,*

AB$=$b$-$a.

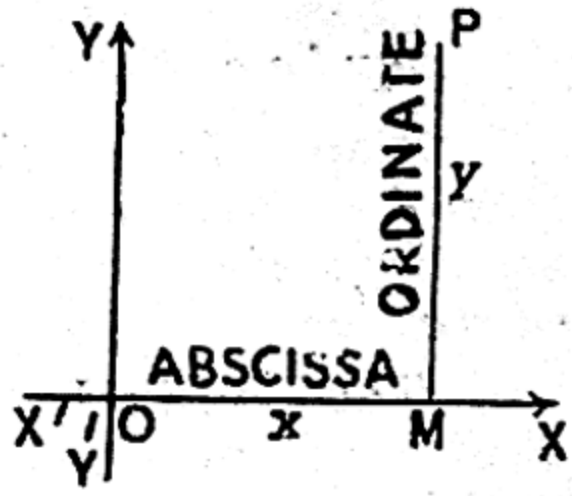
For, AB$=$OB$-$OA$=b-a$.

3. Rectangular axes. Defs. Let X′OX, Y′OY be two directed lines meeting at right angles in O. Then (*i*) X′OX is called the *axis of x* or, more shortly, the **x-axis,** (*ii*) Y′OY the *axis of y* or, more shortly, the **y-axis**, and (*iii*) the two together are called the **rectangular‡ axes** ; (*iv*) O is called the **origin.**



Co-ordinates of a point

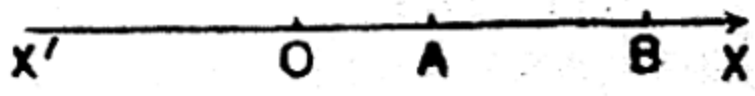
4. (*a*) **Co-ordinates. Def. and notation.** Let P be a point in the plane of the axes. From P draw PM $\perp$ on the *x*-axis. Then (*i*) the length OM is called the **abscissa** of P and is denoted by *x*, (*ii*) the length MP is called the **ordinate** of P and is denoted by *y*, and (*iii*) the two together are called the **co-ordinates** of P and are denoted by (*x*, *y*).



Note. They are also called the **Cartesian co-ordinates** after *Cartesius*, the latinised name of their inventor Des Cartes [pronounced "De Kahrt" (*ah*—very short)]. See also the frontispiece.



***Illustration.** In the adjoining Fig. AB,



the length measured from A to B (*i.e.*, to the right)$=+2$, and

BA, the length measured from B to A (*i.e.*, to the left)$=-2$.

Thus AB$=-$BA.

†**Why this step.** In order to prove that AB$=$OB$-$OA (*i.e.*, by transposing negative term from one side to the other) we have to prove that OA$+$AB$=$OB,

$\therefore$ we consider OA$+$AB.

‡So called, because they are *at right angles.*

(*b*) **Rule for signs of co-ordinates.**

(*i*) *x, or the abscissa, is positive if it is measured to the right of the origin (as in the Fig. of Art. 4, (a)), and negative if measured to the left.*

(*ii*) *y, or the ordinate, is positive if it is measured above the x-axis (as in the Fig. of Art. 4, (a)), and negative if measured below.*

[**Aid to memory.** "Plus to the right ; Minus to left ;
Positive, height ; Negative, depth". —*Briggs and Bryan.*]

Note that the above Rule for signs of co-ordinates is the same as in Trigonometry, the Rule for signs of the sides OM, MP of the triangle OMP. See the author's *New Intermediate Plane Trigonometry (6th Edition) Art. 16, (b).*

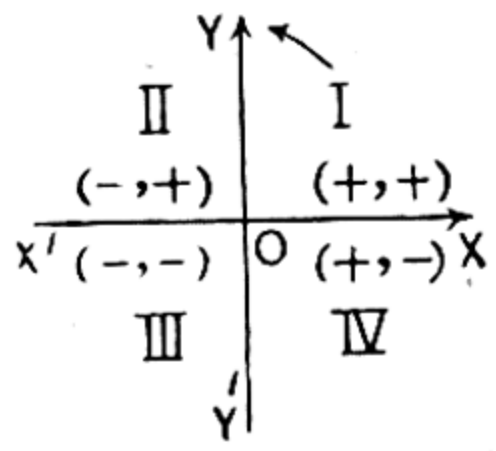
Important. *In naming the co-ordinates of a point, the abscissa is stated* **first** *and then the ordinate, they are separated by a comma and written within brackets thus :* (x, y).

Point (x, y). Def. *The point whose co-ordinates are* (x, y) *is, for shortness, called the* **point (x, y)** or simply **(x, y)**. Thus the origin is **(0, 0)**.

Notation. When the co-ordinates of a point are known they are denoted (*i*) by the earlier letters of the alphabet as (a, b), (c, d), (h, k) ; etc., or (*ii*) by attaching suffixes to x and y as (x_1, y_1) [read "x one, y one"], (x_2, y_2) [read "x two, y two"] ; etc., or (*iii*) by attaching dashes to x and y as (x', y') [read "x dash, y dash"], (x'', y'') [read "x double dash, y double dash"] ; etc.

When the co-ordinates of a point are not known they are denoted, as in Algebra, by the letters x, y as (x, y).

(*c*) **Quadrants. Def.** The axes divide the plane into four parts called **quadrants**. (*i*) XOY is called the **first quadrant,** (*ii*) YOX′ the **second**, (*iii*) X′OY′ the **third,** and (*iv*) Y′OX the **fourth** as marked in the Fig. in which the proper signs of the co-ordinates are also indicated.



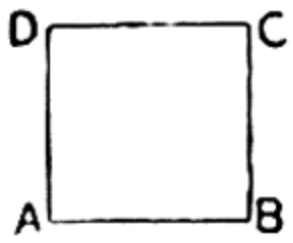
[**Aid to memory.** The quadrants are numbered **I, II, III, IV** in the order in which a revolving line, starting from OX, and revolving in the *positive* (*i.e., counter-clockwise*) direction, passes through the four quadrants.]

EXAMPLE

Find the co-ordinates of the vertices of the square ABCD (side $=2a$)

(*i*) taking AB, AD as axes ;

(*ii*) taking the centre of the square as origin and axes parallel to the sides AB, AD.

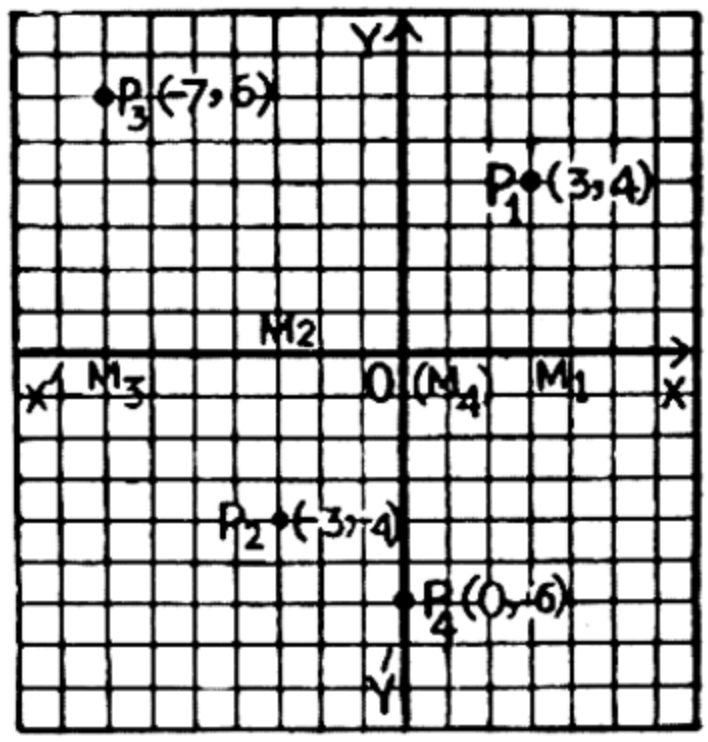


(*d*) **To plot a point whose co-ordinates are given.**

Example. *Plot the points* (*i*) $(3, 4)$; (*ii*) $(-3, -4)$; (*iii*) $(-7, 6)$; (*iv*) $(0, -6)$.

[**Scale.** 1 small division=1.]

(*i*) To plot (3, 4). Measure OM_1 along the x-axis, to the right of O and =3, measure $M_1P_1 \perp$ to the x-axis, above it and =4. Then P_1 is the required pt.



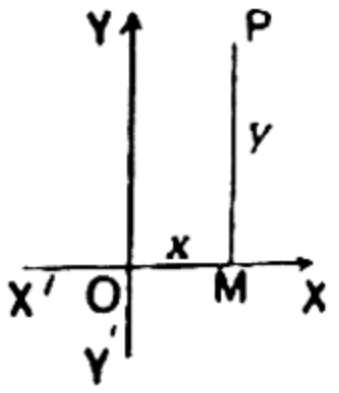
(*ii*) To plot (−3, −4). Measure OM_2 along the x-axis, to the *left* of O and =3 in magnitude, measure $M_2P_2 \perp$ to the x-axis, *below* it and =4 in magnitude. Then P_2 is the required pt.

(*iii*) To plot (−7, 6). Measure OM_3 along the x-axis, to the left of O and =7 in magnitude, measure $M_3P_3 \perp$ to the x-axis, above it and =6. Then P_3 is the required pt.

(*iv*) To plot (0, −6). Measure OM_4 along the x-axis and =0, *i.e.*, at O measure $OP_4 \perp$ to the x-axis, below it and =6 in magnitude. Then P_4 is the required pt.

[Rule to plot the point (x, y) :

Measure OM along the x-axis and =x, measure MP ⊥ to the x-axis and =y, observing the Rule for signs of co-ordinates (Art. 4, (b)). Then P is the required pt.]



EXAMPLE

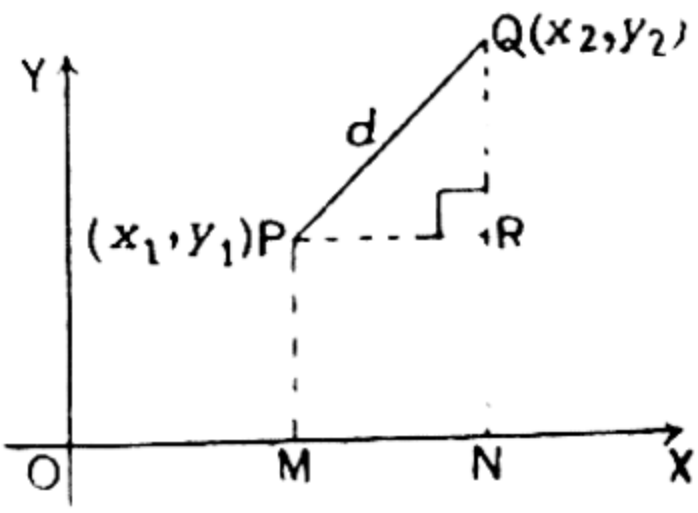
Plot the points (1, 2), (2, −3), (−3, 4), (−4, −5), (−1, 0), (0, 1).

SECTION II

DISTANCE BETWEEN TWO POINTS

5. Distance formula. To find the distance between two points whose co-ordinates are given.*

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be the two pts. Let d be the required distance PQ. From P, Q draw PM, QN $\perp$s on the x-axis and PR $\perp$ on NQ.



Then from the rt. $\angle$d $\triangle$ PRQ,

$$d^2 = PR^2 + RQ^2 \quad \ldots(1)$$

But $PR = MN = ON - OM$

$$= x_2 - x_1,$$

$$RQ = NQ - NR = NQ - MP = y_2 - y_1,$$

$\therefore$ from (1), $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

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*[*P. U. 1951*]

$\therefore$ $\mathbf{d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$.

Note 1. Complete distance formula.

$\because$ $d^2=(x_2-x_1)^2+(y_2-y_1)^2$

$\therefore$ $d=\pm\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$,

that sign being taken on the R. H. S. which gives a +ve* result for d.

Taking +ve sign with the square root,

d, the distance *in magnitude*, $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

Cor. The distance of the point (x_1, y_1) from the origin is $\sqrt{x_1^2+y_1^2}$.

	(1)	(2)
	(0,0)	(x_1,y_1)
For,	$d=\sqrt{(x_1-0)^2+(y_1-0)^2}$	$[\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}]$
	$=\sqrt{x_1^2+y_1^2}$.	

** [**Note 2. Generalisation.** In the Fig. of Art. 5, it has been supposed that P, Q both lie in [redacted] quadrant, but the formula is always true in *whatever* quadrants the points may lie.

[For, in *every* case $PR=MN=ON-OM=x_2-x_1$ (Art. 2, Cor.), and similarly $RQ=y_2-y_1$.]

In fact "*Theorems and formulae deduced by reasoning with points or lines in the first quadrant (where the co-ordinates are always positive) must, from the very nature of the method of Co-ordinate Geometry, hold true when the points or lines are situated in the other quadrants.*

—G. A. Wentworth.

The same proof holds good for every case with due regard to signs of the quantities involved.]

Example. *Find the distance between the points (1, −2) and (−2, 2).*

	(1)	(2)
The distance between the pts. is	(1, −2)	(−2, 2)
$d=\sqrt{(-2-1)^2+[2-(-2)]^2}$	$[\sqrt{x_2-x_1)^2+(y_2-y_1)}]$	

$=\sqrt{(-3)^2+(4)^2}=\sqrt{9+16}=\sqrt{25}=5$.

EXAMPLES

1. (*a*) **Express the distance between two points in terms of their co-ordinates.**

(*b*) Find the distance between the following pairs of points :

(*i*) (−6, 7) and (−1, −5) ; (*ii*) $(-a, a)$ and $(b, -b)$.

2. Find the distance between the following pairs of points :

(*i*) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$;

(*ii*) $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$.

3. Find the value of x_1 if the distance between the points $(x_1, -1)$ and (3, 2) be 5.

4. A line is of length 5 and one end is at the point (1,−2); if the abscissa of the other end is 4, prove that its ordinate must be 2 or −6.

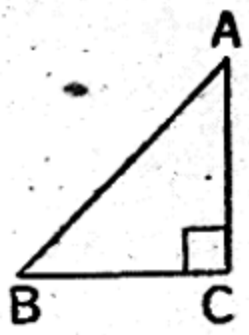
5. Show that the points (a, a), (−a, −a) and $(-a\sqrt{3}, a\sqrt{3})$ are the vertices of an equilateral triangle. [*Pesh. U. 1953*]

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* '+ve' stands for 'positive'. Similarly '−ve' stands for 'negative.'

6. Show that the points (12, 8), (−2, 6) and (6, 0) are the vertices of a right-angled triangle. *[P.U. 1946]*

[**Sol.** Let A(12, 8), B(−2, 6), C(6, 0) be the vertices of the triangle. Then



$AB^2=(-2-12)^2+(6-8)^2=(-14)^2+(-2)^2=196+4=200,$
$BC^2=[6-(-2)]^2+(0-6)^2=(8)^2+(-6)^2=64+36=100,$
$CA^2=(12-6)^2+(8-0)^2=(6)^2+(8)^2=36+64=100,$
$\therefore\ AB^2=BC^2+CA^2$
$\therefore\ \angle ACB=1$ rt. $\angle$. (Converse of Pythagoras' theorem)]

[**Test for a right-angled triangle.**
Square of the *greatest* side
=sum of the squares of the other two sides.]

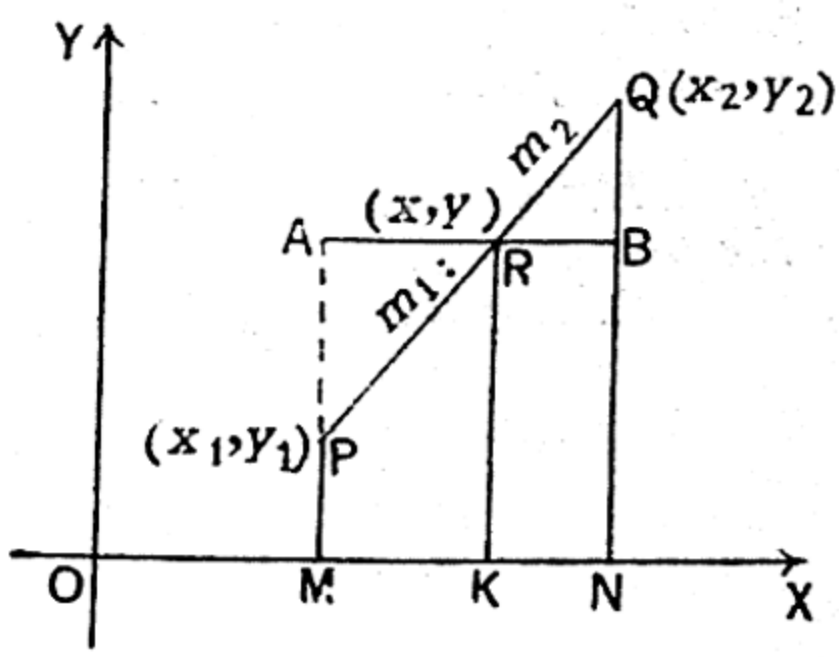
7. Show that the points (0, −1), (2, 1), (0, 3) and (−2, 1) are the corners of a square.

8. Show that (1, −1) is the centre of the circle circumscribing the triangle whose angular points are (4, 3), (−2, 3), and (6, −1).

SECTION III
CO-ORDINATES OF POINTS DIVIDING THE JOIN OF TWO GIVEN POINTS IN A GIVEN RATIO

6-I. Section formulae for internal division. To find the co-ordinates of the point which divides the line joining two given* points internally in a given ratio. †

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be the two pts. Let (x, y) be the required co-ordinates of the pt. R which divides PQ internally in the given ratio $m_1 : m_2$.



From P, Q, R draw PM, QN, RK ⊥s on the x-axis. Thro' R draw a st. line ARB ∥ to the x-axis to meet MP (produced) in A and NQ in B.

Then from the similar △s APR, BQR,

$$\frac{AR}{RB}=\frac{PR}{RQ}=\frac{m_1}{m_2} \quad \dots (1)$$

But $AR=MK=OK-OM=x-x_1,$
$RB=KN=QN-OK=x_2-x,$

$\therefore$ from (1), $\dfrac{x-x_1}{x_2-x}=\dfrac{m_1}{m_2}$ or $m_2x-m_2x_1=m_1x_2-m_1x$

or $x(m_1+m_2)=m_1x_2+m_2x_1$ or $\mathbf{x=\dfrac{m_1x_2+m_2x_1}{m_1+m_2}}$.

Again $\dfrac{PA}{BQ}=\dfrac{PR}{RQ}=\dfrac{m_1}{m_2} \quad \dots (2)$

But $PA=MA-MP=KR-MP=y-y_1,$
$BQ=NQ-NB=NQ-KR=y_2-y.$

***Def. A given point** is a point whose co-ordinates are given.

†[*Pesh. U. 1954*]

$\therefore$ from (2), $\dfrac{y-y_1}{y_2-y}=\dfrac{m_1}{m_2}$ or $m_2y-m_2y_1=m_1y_2-m_1y$

or $y(m_1+m_2)=m_1y_2+m_2y_1$ or $\mathbf{y=\dfrac{m_1y_2+m_2y_1}{m_1+m_2}}$.

[**Note. How to remember the construction and proof of Art. 6-I.** Through the point of division R draw a straight line ARB $\parallel$ to the x-axis to meet the ordinates MP, NQ of the given points P, Q in A,B, and consider the two similar $\triangle$s so formed.

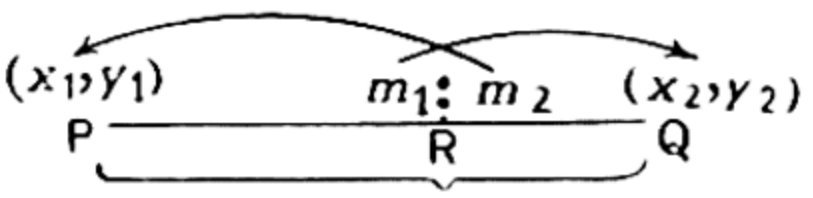
To obtain the x-co-ordinate of R, consider the ratio of the sides AR, RB (of the two similar $\triangle$s) which are $\parallel$ to the x-axis.

Again. to obtain the y-co-ordinate of R, consider the ratio of the sides PA, BQ which are $\parallel$ to the y-axis.

The same construction and proof will be adopted in Art. 6-II.]

[**Rule to write down the co-ordinates of the point which divides the join of two given points internally in a given ratio $m_1 : m_2$.**

(*i*) *Multiply m_1 by the x-co-ordinate of the point remote from m_1 and m_2 by the x-co-ordinate of the point remote from m_2 as shown by the arrows,*



(ii) add these products, and (iii) divide the sum by m_1+m_2.

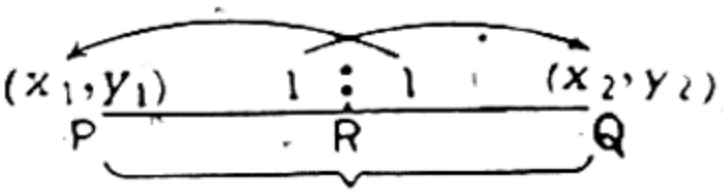
This gives the x-co-ordinate of the point of division.

Similarly for the y-co-ordinate.

Cor. Mid-point formulae. *The co-ordinates of the mid-point of the join of (x_1, y_1) and (x_2, y_2) are*

$$\mathbf{x=\frac{x_1+x_2}{2}}, \qquad \mathbf{y=\frac{y_1+y_2}{2}}.$$

For, let P, Q be the pts. (x_1, y_1), (x_2, y_2), and (x, y) the required co-ordinates of R, the mid-pt. of PQ.



Then $PR=RQ$ or $\dfrac{PR}{RQ}=\dfrac{1}{1}$

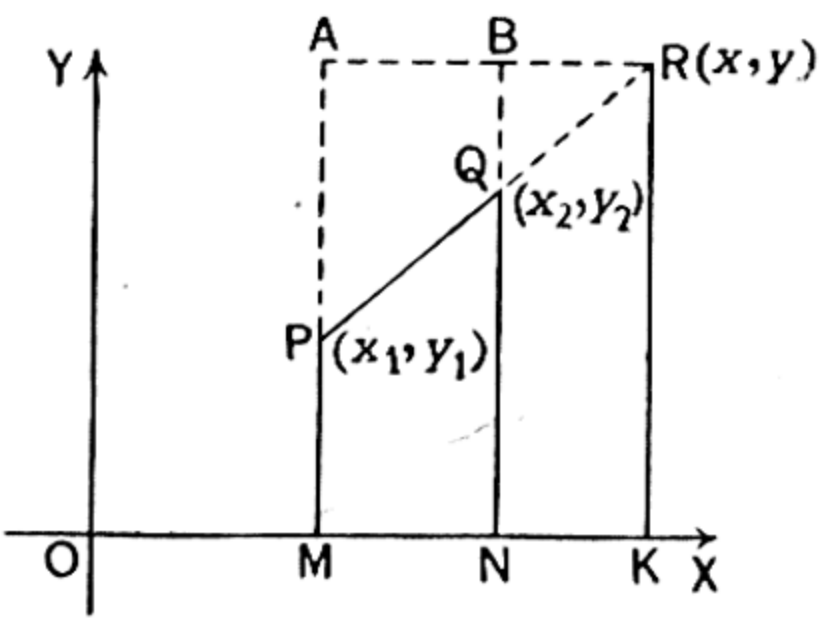
$$\therefore \qquad \mathbf{x=\frac{1.x_2+1.x_1}{1+1}=\frac{x_1+x_2}{2}}, \quad \mathbf{y=\frac{1.y_2+1.y_1}{1+1}=\frac{y_1+y_2}{2}}.$$

6-II. Section formulae for external division. To find the co-ordinates of the point which divides the line joining two given points externally in a given ratio. *

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be the two pts. Let (x, y) be the required co-ordinates of the pt. R which divides PQ *externally* in the given ratio $m_1 : m_2$

From P, Q, R draw PM, QN, RK $\perp$s on the x-axis. Thro' R draw a st. line ABR $\parallel$ to the x-axis to meet MP and NQ both produced in A and B.

Then from the similar $\triangle$s APR, BQR,



* [*J. & K. U. 1952*]

$$\frac{AR}{BR}=\frac{PR}{QR}=\frac{m_1}{m_2} \quad \dots (1)$$

But $AR = MK=OK-OM=x-x_1,$

$BR = NK=OK-ON=x-x_2,$

$\therefore$ from (1), $\frac{x-x_1}{x-x_2}=\frac{m_1}{m_2}$ or $m_2x-m_2x_1=m_1x-m_1x_2$

or $x(m_2-m_1)=m_2x_1-m_1x_2$

$\therefore$ $$\mathbf{x}=\frac{m_2x_1-m_1x_2}{m_2-m_1}=\frac{\mathbf{m_1x_2-m_2x_1}}{\mathbf{m_1-m_2}}.$$

Again $$\frac{PA}{QB}=\frac{PR}{QR}=\frac{m_1}{m_2} \quad \dots (2)$$

But $PA = MA-MP=KR-MP=y-y_1,$

$QB = NB-NQ=KR-NQ=y-y_2,$

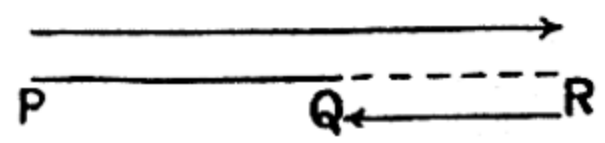
$\therefore$ from (2), $\frac{y-y_1}{y-y_2}=\frac{m_1}{m_2}$ or $m_2y-m_2y_1=m_1y-m_1y_2$

or $y(m_2-m_1)=m_2y_1-m_1y_2$

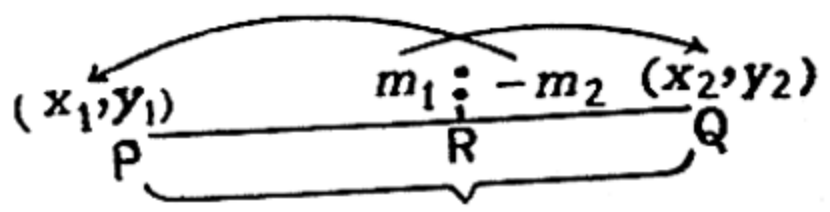
$\therefore$ $$\mathbf{y}=\frac{m_2y_1-m_1y_2}{m_2-m_1}=\frac{\mathbf{m_1y_2-m_2y_1}}{\mathbf{m_1-m_2}}.$$

Another method. *To deduce the section formulae for external division from those for internal division.*

Here $\frac{PR}{QR}=\frac{m_1}{m_2}$, but $QR=-RQ$ [Art. 2, (1)]



$\therefore$ $\frac{PR}{-RQ}=\frac{m_1}{m_2}$ or $\frac{PR}{RQ}=\frac{m_1}{-m_2}$



$\therefore$ the co-ordinates of R are

$$x=\frac{m_1x_2+(-m_2)x_1}{m_1+(-m_2)}=\frac{m_1x_2-m_2x_1}{m_1-m_2},$$

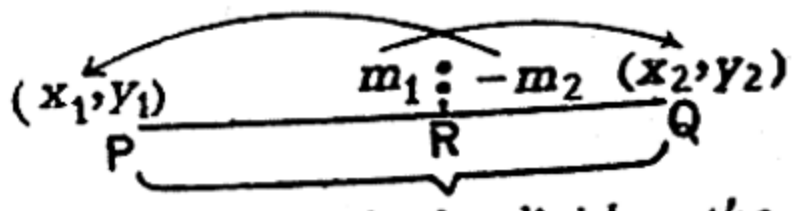
$$y=\frac{m_1y_2+(-m_2)y_1}{m_1+(-m_2)}=\frac{m_1y_2-m_2y_1}{m_1-m_2}.$$

Note. If R divides PQ *internally* as in the Fig. of Art. 6-I, then the ratio $\frac{PR}{RQ}$ is **positive**. [$\because$ PR and RQ have the same sign (direction)]

But if R divides PQ *externally* as in the Fig. of Art. 6-II, then the ratio $\frac{PR}{RQ}$ is **negative**. [$\because$ PR and RQ have opposite signs]

[Rule to write down the co-ordinates of the point which divides the join of two given points *externally* in a given ratio $m_1 : m_2$.

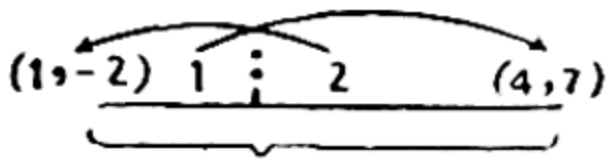
Change m_2 to $-m_2$ and write down the co-ordinates of the point which divides the join of the given points **internally** *in the ratio $m_1 : -m_2$.* [*Art. 6-I, Rule*]



Example 1. *Find the co-ordinates of the point which divides the line joining (1, −2) and (4, 7)*

(*i*) *internally in the ratio 1 : 2 ;* (*ii*) *externally in the ratio 2 : 3.*

(*i*) The co-ordinates of the pt. of division are



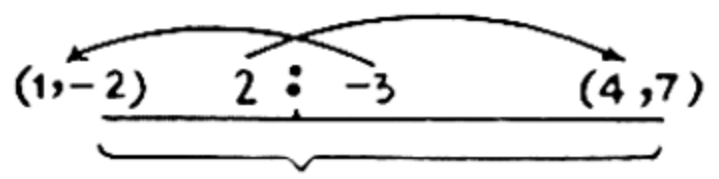
$$x = \frac{1(4)+2(1)}{1+2} \quad \text{[Rule (Art. 6-I)]}$$

$$= \frac{6}{3} = 2,$$

$$y = \frac{1(7)+2(-2)}{1+2} = \frac{3}{3} = 1.$$

$\therefore$ the co-ordinates of the pt. of division are (2, 1).

(*ii*) The co-ordinates of the pt. of division are



$$x = \frac{2(4)+(-3)(1)}{2+(-3)} \quad \text{[Rule (Art. 6-II)]}$$

$$= \frac{5}{-1} = -5,$$

$$y = \frac{2(7)+(-3)(-2)}{2+(-3)} = \frac{20}{-1} = -20.$$

$\therefore$ the co-ordinates of the pt. of division are (−5, −20).

Example 2. *Find the middle point of the line joining* (−3, −6) *and* (1, −2).

The co-ordinates of the mid-pt. are $\overset{(1)}{(-3, -6)}, \overset{(2)}{(1, -2)}$

$$x = \frac{(-3)+1}{2} \qquad \left[\frac{x_1+x_2}{2}\right]$$

$$= \frac{-2}{2} = -1,$$

$$y = \frac{(-6)+(-2)}{2} \qquad \left[\frac{y_1+y_2}{2}\right]$$

$$= \frac{-8}{2} = -4.$$

$\therefore$ the mid-pt. is (−1, −4).

EXAMPLES 2

1. (*a*) **Find the co-ordinates of a point which divides in a given ratio the straight line joining two given points.** [*P(P). U. 1949*]

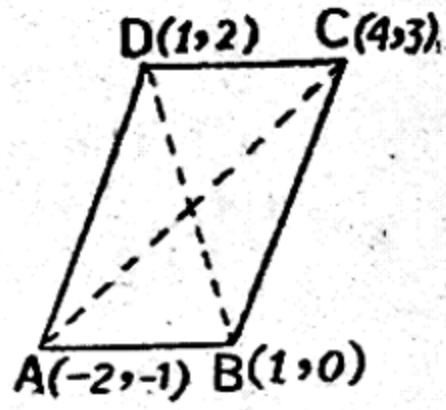
(*b*) **Find the co-ordinates of the points which divide internally and externally, the line joining (1, −3) and (−3, 9) in the ratio 1 : 3.**

[**Note.** (*a*) **Important.** "*Divides in a given ratio*" means "divides in a given ratio (*i*) *internally*, and (*ii*) *externally*." So both the Arts. 6-I and 6-II should be reproduced.]

2. (*a*) **Find the co-ordinates of the point which divides the line joining the two points (a_1, b_1) and (a_2, b_2) internally in the ratio l : m.** [*P(P). U. 1955*]

(*b*) **Prove that the points (−2, −1), (1, 0), (4, 3) and (1, 2) are the vertices of a parallelogram.** [*Mysore U.*]

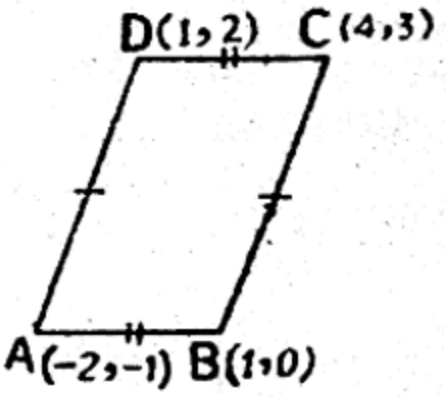
[**Sol.** (*b*) **First method.** Let A(−2, −1), B(1, 0), C(4, 3), D(1, 2) be the pts. Then the mid-pt. of the diagonal AC is $\left(\frac{-2+4}{2}, \frac{-1+3}{2}\right)$, $\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right]$



i.e., (1, 1),
and the mid-pt. of the diagonal BD is
$\left(\frac{1+1}{2}, \frac{0+2}{2}\right)$, *i.e.*, (1, 1).

$\therefore$ the mid-pt. of AC is the same as the mid-pt. of BD,
$\therefore$ ABCD is a ||gm. ($\because$ diagonals bisect one another)

Second method. Let A(−2, −1), B(1, 0), C(4, 3), D(1, 2) be the pts.



Then $AB=\sqrt{[1-(-2)]^2+[0-(-1)]^2}$
$=\sqrt{(3)^2+(1)^2}=\sqrt{9+1}=\sqrt{10}$,
$DC=\sqrt{(4-1)^2+(3-2)^2}$
$=\sqrt{(3)^2+(1)^2}=\sqrt{9+1}=\sqrt{10}$,
$AD=\sqrt{[1-(-2)]^2+[2-(-1)]^2}=\sqrt{(3)^2+(3)^2}=\sqrt{9+9}=\sqrt{18}$
$=3\sqrt{2}$,
$BC=\sqrt{(4-1)^2+(3-0)^2}=\sqrt{(3)^2+(3)^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$.
$\therefore$ AB=DC, AD=BC,
$\therefore$ ABCD is a ||gm. ($\because$ opposite sides are equal)]

[**Test for a parallelogram.** *The mid-point of one diagonal is the same as the mid-point of the other diagonal.*
Or: *The opposite sides are equal.*]

3. Show that the four points (1, 2), (3, 0), (7, 4) and (5, 6) are the angular points of a parallelogram.

4. Show that the points (1, −1), (−2, 2), (4, 8) and (7, 5) are at the angular points of a rectangle.

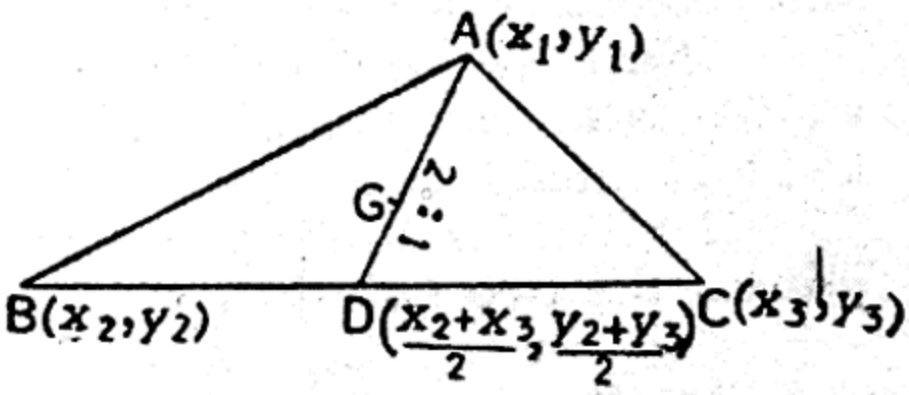
Centroid and incentre of a triangle

7. (*a*) **Centroid. Def. and notation.** *The point of concurrence of the medians of a triangle is called the* **centroid** *of the triangle. It divides each median in the ratio 2 : 1 (from Geometry) and is generally denoted by G.*

Example 1. Centroid formulae. If (x_1, y_1), (x_2, y_2), (x_3, y_3) are the vertices of a triangle, the co-ordinates of its centroid are

$$x=\frac{x_1+x_2+x_3}{3},\quad y=\frac{y_1+y_2+y_3}{3}.$$ [*P (P). U. 1955*]

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle.



Then D, the mid-pt. of BC, is
$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$.

$\therefore$ the co-ordinates of the *centroid G which divides AD internally in the ratio 2 : 1*,

are $$\mathbf{x}=\frac{2\left(\frac{x_2+x_3}{2}\right)+1(x_1)}{2+1}=\frac{\mathbf{x_1+x_2+x_3}}{3},$$

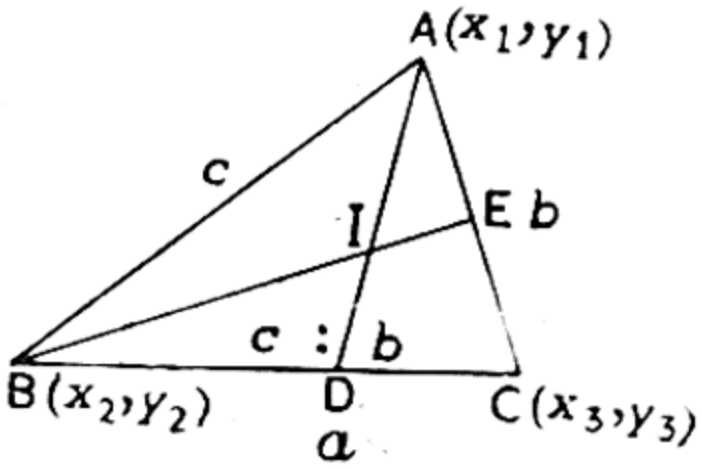
$$\mathbf{y}=\frac{2\left(\frac{y_2+y_3}{2}\right)+1(y_1)}{2+1}=\frac{\mathbf{y_1+y_2+y_3}}{3}.$$

(*b*) **Incentre. Def. and notation.** *The point of concurrence of the internal bisectors of the angles of a triangle is called the* **incentre** *of the triangle. It is generally denoted by I.*

Example 2. Incentre formulae. The co-ordinates of the vertices of a triangle being given, find the co-ordinates of its incentre.

[J. & K. U. 1952]

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle. Let AD and BE, the bisectors of the $\angle$s A and B, meet in I. Then I is the incentre of the triangle.



[Rule (Art. 6-I)]

$\because$ AD bisects $\angle$BAC of $\triangle$ ABC

$\therefore$ by Geometry, $\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$,

i.e., D divides BC internally in the ratio $c : b$.

$\therefore$ D is $\left(\frac{cx_3+bx_2}{c+b}, \frac{cy_3+by_2}{c+b}\right)$.

Now BI bisects $\angle$ABD of $\triangle$ ABD

$\therefore \frac{AI}{ID}=\frac{AB}{BD}=\frac{c}{BD}$ ...(1) [To find BD]

But $\frac{BD}{DC}=\frac{c}{b}$.[Proved above]

$\therefore \frac{BD}{BD+DC}=\frac{c}{c+b}$ [Componendo]

or $\frac{BD}{BC}=\frac{c}{c+b}$, $\therefore$ $BD=\frac{ca}{c+b}$ [$\because$ $BC=a$]

$\therefore$ from (1), $\frac{AI}{ID}=\frac{c}{\frac{ca}{c+b}}=\frac{c+b}{a}$, *i.e.*, I divides AD internally in the ratio $c+b : a$.

$\therefore$ the co-ordinates of I are

$$\mathbf{x}=\frac{(c+b)\left(\frac{cx_3+bx_2}{c+b}\right)+ax_1}{(c+b)+a}=\frac{\mathbf{ax_1+bx_2+cx_3}}{\mathbf{a+b+c}},$$

$$\mathbf{y}=\frac{(c+b)\left(\frac{cy_3+by_2}{c+b}\right)+ay_1}{(c+b)+a}=\frac{\mathbf{ay_1+by_2+cy_3}}{\mathbf{a+b+c}}.$$

EXAMPLES 3

1. Find the centroid of the triangle whose angular points are (3, −5), (−7, 4) and (10, −2) respectively. *[P. U. 1939S]*

[**Sol.** The vertices of the triangle are (3, −5), (−7, 4), (10, −2).
∴ the co-ordinates of the centroid are

$$x=\frac{3+(-7)+10}{3} \quad \cdot\left[\frac{x_1+x_2+x_3}{3}\right] \quad \text{[Solved Ex. 1, Art. 7]}$$

$$=\frac{6}{3}=2,$$

$$y=\frac{(-5)+4+(-2)}{3} \quad \left[\frac{y_1+y_2+y_3}{3}\right]$$

$$=\frac{-3}{3}=-1.$$

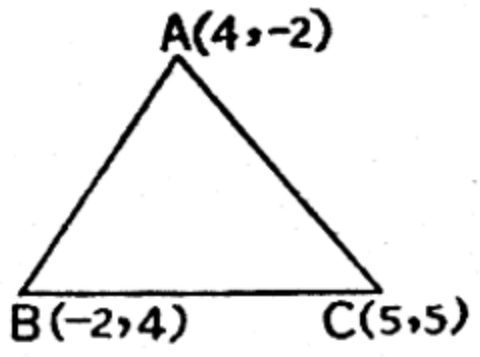
∴ the centroid is (2, −1).]

2. Find the centroid of the triangle whose angular points are (−4, 6), (2, −2) and (2, 5) respectively. *[D. U. 1946]*

3. Prove that the medians of a triangle are concurrent. *[Pesh. U. 1954]*

4. Find the co-ordinates of the centre of the circle inscribed in the triangle whose angular points are (4, −2), (−2, 4) and (5, 5). *[P. U.]*

[**Sol.** Let A(4, −2), B(−2, 4), C(5, 5) be the vertices of the triangle.



Then $a=BC=\sqrt{[5-(-2)]^2+(5-4)^2}$

$=\sqrt{(7)^2+(1)^2}=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$, *(Note this step)*

$b=CA=\sqrt{(4-5)^2+(-2-5)^2}$

$=\sqrt{(-1)^2+(-7)^2}=\sqrt{1+49}=\sqrt{50}=5\sqrt{2}$,

$c=AB=\sqrt{(-2-4)^2+[4-(-2)]^2}=\sqrt{(-6)^2+(6)^2}$

$=\sqrt{36+36}=\sqrt{72}=6\sqrt{2}$.

∴ the co-ordinates of the incentre are

$$x=\frac{ax_1+bx_2+cx_3}{a+b+c}=\frac{5\sqrt{2}(4)+5\sqrt{2}(-2)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}=\frac{40\sqrt{2}}{16\sqrt{2}}=\frac{5}{2},$$

$$y=\frac{ay_1+by_2+cy_3}{a+b+c}=\frac{5\sqrt{2}(-2)+5\sqrt{2}(4)+6\sqrt{2}(5)}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}=\frac{40\sqrt{2}}{16\sqrt{2}}=\frac{5}{2}.$$

∴ the co-ordinates of the incentre are $\left(\frac{5}{2}, \frac{5}{2}\right)$.]

5. Find the co-ordinates of the incentre of the triangle whose vertices are (−36, 7), (20, 7) and (0, −8). *[P. U. 1950]*

6. Prove that the internal bisectors of the angles of a triangle are concurrent. *[A. U. 1946]*

8. Choice of axes. In order to prove a geometrical theorem when the co-ordinates of points are not given, the choice of axes is at our disposal, and a judicious choice simplifies the algebraic work considerably.

Two Rules for the choice of axes

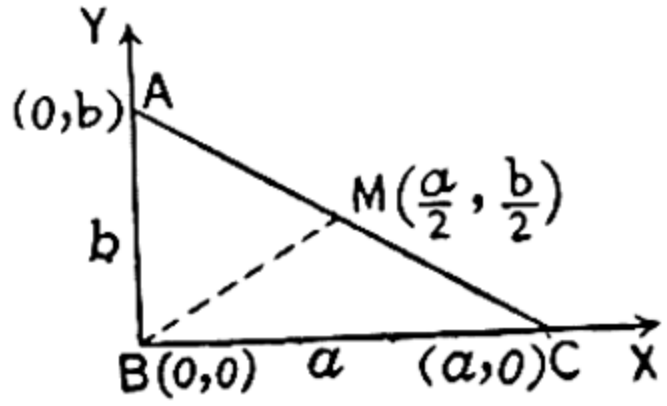
8-I. Rule I. For problems relating to two given perpendicular lines, take these lines as the axes.

Thus for problems relating to a right-angled triangle or a rectangle (or a square) we may take the perpendicular sides as the axes.

Note. In the case of a rectangle (or a square) we may take, sometimes with advantage, the centre as origin and the lines through it parallel to the sides as the axes.

Example. *Prove analytically* that in a right-angled triangle the mid-point of the hypotenuse is equidistant from the three angular points.* [*P. U. 1948 Em.*]

Let ABC be the triangle rt. ∠d at B.
Take BC, BA as the axes.



Let BC=a, BA=b.

Then B is (0, 0), C is (a, 0), A is (0, b).

M, the mid-pt. of AC, is $\left(\frac{0+a}{2}, \frac{b+0}{2}\right)$

i.e., $\left(\frac{a}{2}, \frac{b}{2}\right)$.

$$\therefore \quad AM=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-b\right)^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2}$$

$$=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\frac{\sqrt{a^2+b^2}}{2},$$

$$BM=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-0\right)^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}$$

$$=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\frac{\sqrt{a^2+b^2}}{2},$$

$$CM=\sqrt{\left(\frac{a}{2}-a\right)^2+\left(\frac{b}{2}-0\right)^2}=\sqrt{\left(-\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}$$

$$=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\frac{\sqrt{a^2+b^2}}{2}.$$

$\therefore$ AM=BM=CM.

***Note.** "*Analytically*" *means "by Co-ordinate Geometry."*

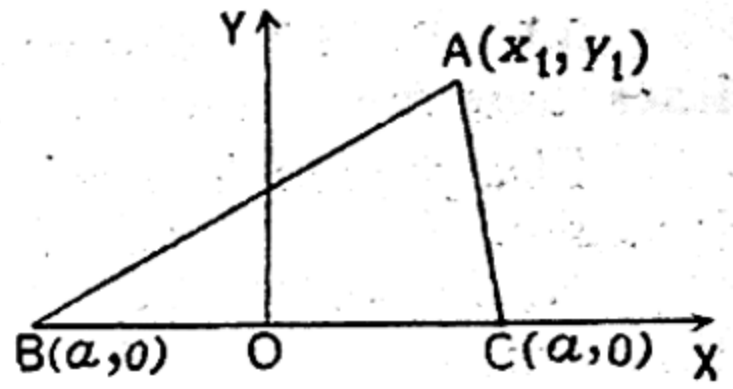
EXAMPLE

Prove analytically that the diagonals of a rectangle are equal.

8-II. Rule II. For problems relating to two given points, take the mid-point of their join as the origin and the join as the x-axis, so that the line through the mid-point perpendicular to the join is the y-axis.

Then, if the join=2a, the co-ordinates of the given points are (a, 0) and (−a, 0).

Thus for problems relating to any triangle we may take the mid-point of the base as the origin and the base as the x-axis, so that the line through the mid-point perpendicular to the base is the y-axis.



Then, if the base $=2a$, the co-ordinates of the extremities of the base are $(a, 0)$ and $(-a, 0)$, and the co-ordinates of the opposite vertex may be taken as (x_1, y_1).

EXAMPLES 4

1. Median theorem (or Apollonius' theorem*). In any triangle ABC prove that $AB^2+AC^2=2(AD^2+DC^2)$, where D is the middle point of BC. [*D. U. 1953*]

[**Hint.** Take D as origin, DC as the x-axis, so that DY, the line thro' D $\perp$ to DC is the y-axis. Let $BC=2a$. Then D is $(0, 0)$, C is $(a,0)$, B is $(-a,0)$, and let A be (x_1, y_1). $AB^2+AC^2=$?, $2(AD^2+DC^2)=$?]

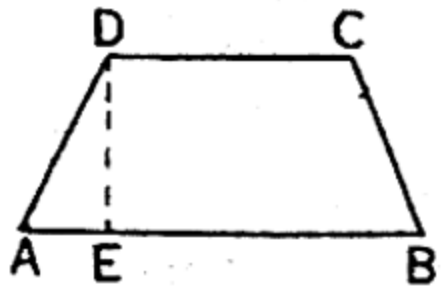
2. *If G is the centroid of a triangle ABC, prove that*

$$3(GA^2+GB^2+GC^2)=AB^2+BC^2+CA^2.$$ [*D.U. 1953*]

SECTION IV

AREA OF A TRIANGLE

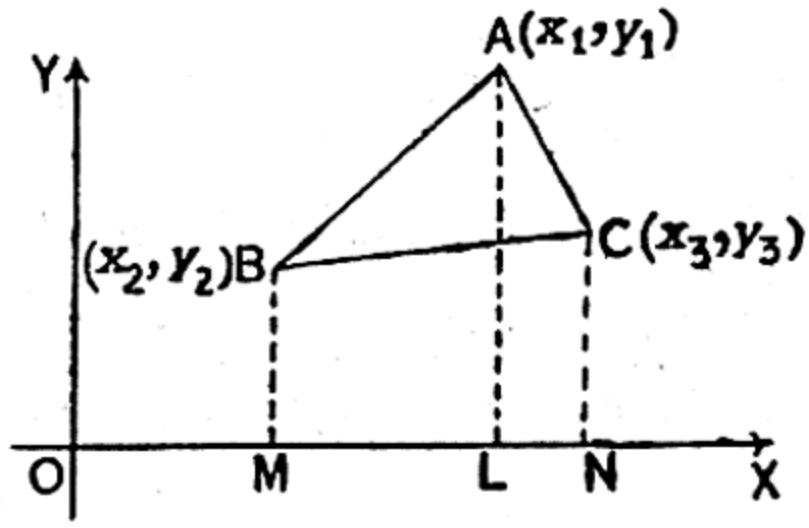
[**Note on the area of a trapezium.**
Let ABCD be a trapezium in which AB ∥ DC.
From D draw DE $\perp$ AB. Then from Geometry,
area of trapezium ABCD $=\frac{1}{2}$(AB+DC). DE.



In words : **Area of a trapezium**
$=\frac{1}{2}$**(sum of ∥ sides)** $\times$ **(⊥ distance between them).**]

9. Area formula. To find the area of a triangle whose vertices are given.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle. Let $\triangle$† be the required area of the triangle. From A,B,C draw AL, BM, CN $\perp$s on the x-axis.



Then $\triangle$ = trapezium ABML + trap. ‡‡ ALNC − trap. BMNC

$$=\tfrac{1}{2}(MB+LA)ML+\tfrac{1}{2}(LA+NC)LN-\tfrac{1}{2}(MB+NC)MN$$
$$=\tfrac{1}{2}(y_2+y_1)(x_1-x_2)+\tfrac{1}{2}(y_1+y_3)(x_3-x_1)-\tfrac{1}{2}(y_2+y_3)(x_3-x_2)$$
$$=\tfrac{1}{2}[x_1y_2-x_2y_2+x_1y_1-x_2y_1+x_3y_1-x_1y_1+x_3y_3-x_1y_3$$
$$-(x_3y_2-x_2y_2+x_3y_3-x_2y_3)]$$

---

*After the discoverer, **Apollonius** (260–200 B.C.) who studied and probably lectured at Alexandria. He got the title of "Great Geometer".

*[*P (P). U. 1956 S*]

†Pronounced *delta*.

‡‡'*trap.*' stands for ' trapezium '.

$$= \tfrac{1}{2}[x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3]$$

$$\therefore \ \triangle = \tfrac{1}{2}[(\mathbf{x_1y_2 - x_2y_1}) + (\mathbf{x_2y_3 - x_3y_2}) + (\mathbf{x_3y_1 - x_1y_3})] \quad \text{... } \textit{First form}$$

$$\text{or } \triangle = \tfrac{1}{2}[\mathbf{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)}] \quad \text{... } \textit{Second form}$$

[Rule to write down the area of a triangle whose vertices are given :

(*i*) *Write down the co-ordinates of the vertices in order in two columns repeating at the end the co-ordinates at the head,* (*ii*) *draw diagonals, and* (*iii*) *prefix the factor* $\frac{1}{2}$ *thus getting*

$$\tfrac{1}{2}\begin{vmatrix} x_1 & \times & y_1 \\ x_2 & \times & y_2 \\ x_3 & \times & y_3 \\ x_1 & & y_1 \end{vmatrix}.$$

Simplification. *Take the products of each row and the next cross-ways putting the minus sign between them* $\left[\text{e.g., } \begin{matrix} x_1 \times y_1 \\ x_2 \times y_2 \end{matrix} = x_1y_2 - x_2y_1 \right]$ *; add these results and multiply the sum by* $\frac{1}{2}$*, thus getting*

$$\triangle = \tfrac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3].$$

****Note 1.** The above Rule enables us to write down the area of a quadrilateral (in fact of a polygon) whose vertices are given in order.

Thus the area of the quadrilateral whose vertices, taken in order, are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)

$$= \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4).]$$

$$\tfrac{1}{2}\begin{vmatrix} x_1 & \times & y_1 \\ x_2 & \times & y_2 \\ x_3 & \times & y_3 \\ x_4 & \times & y_4 \\ x_1 & & y_1 \end{vmatrix}$$

(See Misc. Exs. 18, 19, Chap. I.)

Cor. The area of the triangle whose vertices are the origin and the points (x_1, y_1), (x_2, y_2) is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

For, by the Rule,

$$\triangle = \tfrac{1}{2}[0 - 0 + x_1y_2 - x_2y_1 + 0 - 0] = \tfrac{1}{2}[x_1y_2 - x_2y_1].$$

$$\tfrac{1}{2}\begin{vmatrix} 0 & \times & 0 \\ x_1 & \times & y_1 \\ x_2 & \times & y_2 \\ 0 & \times & 0 \end{vmatrix}$$

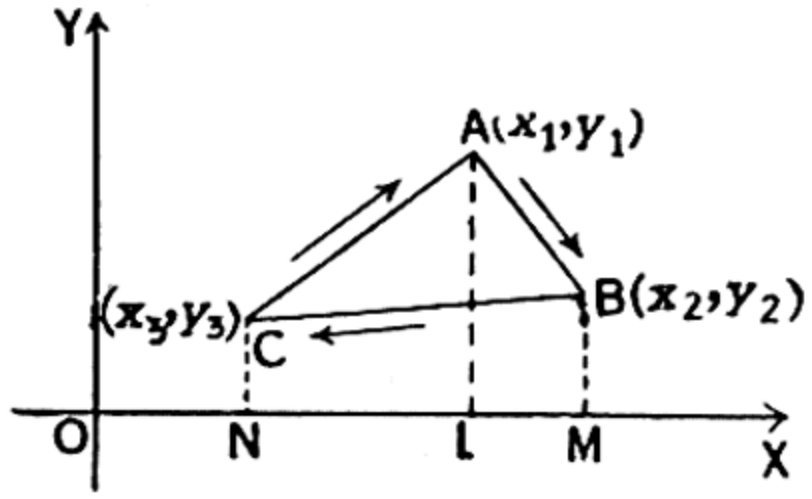
****Note 2. Complete area formula.** If, in going round the triangle in the given order of the vertices, the area (instead of being on the left hand, as in the Fig. of Art. 9) is on the *right* hand, as in the adjoining Fig., it may be proved, by proceeding as in Art. 9, that

$$\triangle = -\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$$

∴ combining this with the area formula of Art. 9,

$$\triangle = \pm\tfrac{1}{2}[(\mathbf{x_1y_2 - x_2y_1}) + (\mathbf{x_2y_3 - x_3y_2}) + (\mathbf{x_3y_1 - x_1y_3})],$$

that sign being taken on the R. H. S. which gives a +ve result for $\triangle$.



****Note 3. Interpretation of the positive and negative results in the area formula.**

In any numerical example,

(*i*) if the area formula $\triangle = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$ gives a positive result, it means that (see Fig. of Art. 9) plotting the vertices of the triangle and going round the triangle in the given order of the vertices, the area is on the left hand ; but

(*ii*) if the area formula gives a negative result, it means that (see Fig. of Note 2, Art. 9) plotting the vertices of the triangle and going round the triangle in the given order of the vertices, the area is on the right hand and we must *change the sign to get the area in magnitude.*

(*Note this step*)

For, from Art. 9,

(*i*) $\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)] = \triangle$,

and from Note 2, Art. 9,

(*ii*) $\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) = -\triangle$.

****Note 4. When to use the complete area formula.** The complete area formula is used when the area of the triangle is *given.* [See Misc. Ex. 16, Chap. I.]

Example. *Find the area of the triangle whose vertices are* (*i*) *(2, 9), (−2, 1) and (6, 3) ;* (*ii*) *(3, 8), (7, 2) and (−1, 1).* [*Pesh. U.*]

(*i*) The area of the triangle is

$$\triangle = \tfrac{1}{2}[2 - (-18) - 6 - 6 + 54 - 6] = \tfrac{1}{2}[56] = 28.$$

$\frac{1}{2}$	2	9
	-2	1
	6	3
	2	9

(*ii*) The area of the triangle is

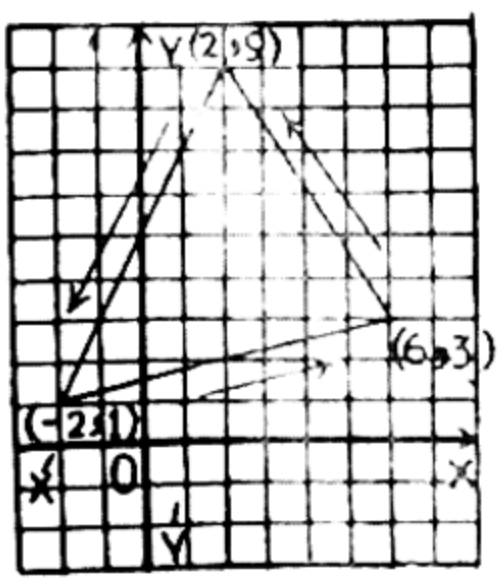
$$\triangle = \tfrac{1}{2}[6 - 56 + 7 - (-2) - 8 - 3] = \tfrac{1}{2}[-52] = -26.$$

$\frac{1}{2}$	3	8
	7	2
	-1	1
	3	8

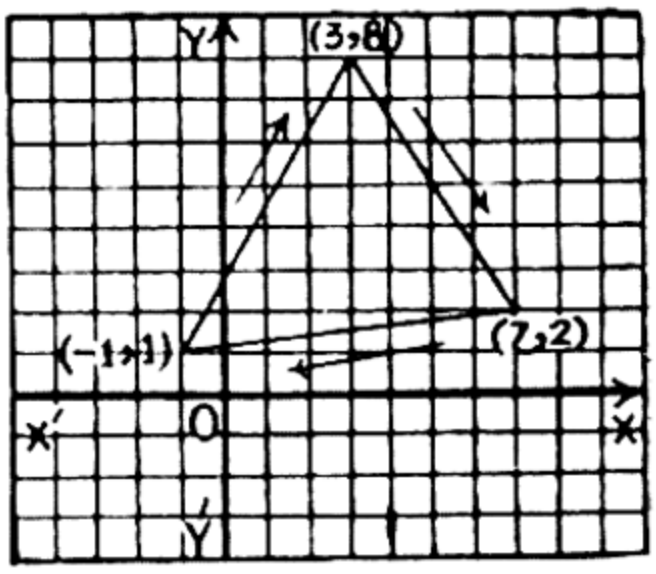
$\therefore$ *changing the sign,*

the area of the triangle *in magnitude* = 26.

****Interpretation of the positive and negative results.**



I



II

Plotting the vertices of the triangle and going round the triangle in the *given* order of the vertices,

in (*i*) the area is on the *left* hand (Fig. I) ; and

in (*ii*) the area is on the *right* hand (Fig. II).

EXAMPLES

1. (*a*) **Express the area of a triangle in terms of the co-ordinates of its angular points.** [*D. U. 1947*]

(*b*) *Find the area of the triangle whose angular points are*

(*i*) *(1, 2), (−2, 3) and (−3,−4) ; (ii) (c, a), (c+a, a) and (c−a,−a).*

2. (*a*) **Find the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3).** [*P. U. 1953 S*]

(*b*) **Find the area of the triangle the co-ordinates of whose angular points are**

(*i*) **(a cos α, b sin α), (a cos β, b sin β) and (a cos γ, b sin γ) ;**

(*ii*) **$(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$.**

[**Short-cut.** The area of the triangle is

$\triangle = \frac{1}{2}\Sigma x_1(y_2-y_3)$ $(\because \triangle = \frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$ [Area formula (Second form) (Art. 9)])

$= \frac{1}{2}\Sigma at_1^2(2at_2-2at_3) = a^2\Sigma t_1^2(t_2-t_3)$

$= -a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$ [From Algebra]

$[\because \Sigma a^2(b-c) = -(a-b)(b-c)(c-a)]$

$\therefore$ *changing the sign,*

the area of the triangle *in magnitude* $= a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$.]

****3.** Find the area of the triangle the co-ordinates of whose angular points are

$[at_1t_2, a(t_1+t_2)]$, $[at_2t_3, a(t_2+t_3)]$ and $[at_3t_1, a(t_3+t_1)]$.

4. The co-ordinates of A, B, C are (6, 3), (−3, 5), (4, −2) respectively and P is the point (x, y) ; show that

$$\frac{\triangle PBC}{\triangle ABC} = \frac{x+y-2}{7}.$$ [*J. & K. U. 1951*]

10. Condition of collinearity of three points. To find the condition that the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) may lie in a straight line.*

If the three pts. lie in a st. line, the area of the triangle formed by them is zero.†

$\therefore$ $\frac{1}{2}[x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3] = 0$

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

or $x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3 = 0,$

which is the required condition.

[**Caution.** Do not commit to memory the above condition of collinearity of three points. In every case use the following

Rule to prove that three given points are collinear :

(*i*) *Find* $\triangle$, *the area of the triangle formed by the points, and* (*ii*) *show that* $\triangle = 0$.]

---

*[*P(P). U. 1948*]

†For, let the three pts. A, B, C lie in a st. line. Then

$A(x_1, y_1)$ $B(x_2, y_2)$ $C(x_3, y_3)$

$\triangle$ ABC $= \frac{1}{2}$BC. ($\perp$ from A on BC) $= \frac{1}{2}$BC$\times 0 = 0$.

Example. Show that the three points (1, 4), (3, −2) and (−3, 16) are collinear. [*P. U.*]

The area of the triangle formed by the three pts. is

$$\frac{1}{2}\begin{vmatrix} 1 & 4 \\ 3 & -2 \\ -3 & 16 \\ 1 & 4 \end{vmatrix}$$

$$\triangle = \tfrac{1}{2}[-2-12+48-6-12-16] = 0$$

$\therefore$ the pts. are collinear.

EXAMPLES 6

1. Prove that the points (−5, 1), (5, 5) and (10, 7) lie on a straight line. [*D. U. 1935*]

2. Show that the points (a, b+c), (b, c+a), (c, a+b) are on a straight line. [*D. U. 1951*]

3. If (x, y) be any point in the straight line which passes through (a, 0) and (0, b), prove that $\frac{x}{a}+\frac{y}{b}=1$. [*Pesh. U.*

MISCELLANEOUS EXAMPLES ON CHAPTER I

1. Lay down in a figure the positions of the points (−1, 3) and (2, −1), and prove that the distance between them is 5.

[**Hint.** "Lay down in a Fig. the position of the point" means "plot the point."]

2. An equilateral triangle has one vertex at the point (3, 4) and another at the point (−2, 3). Find the co-ordinates of the third vertex. [*D. U. 1949*]

[**Hint.** Let A(3, 4), B(−2, 3) be the two vertices and (x, y) the required co-ordinates of the third vertex C. Then BC=CA=AB.

$$\therefore \sqrt{(x+2)^2+(y-3)^2} = \sqrt{(x-3)^2+(y-4)^2} = \sqrt{(-2-3)^2+(3-4)^2}$$
$$= \sqrt{(-5)^2+(-1)^2} = \sqrt{25+1} = \sqrt{26},$$

or, squaring,

$$x^2+4x+4+y^2-6y+9 = x^2-6x+9+y^2-8y+16 = 26$$

or $$x^2+y^2+4x-6y+13 = x^2+y^2-6x-8y+25 = 26.$$

From the first and second members, $10x+2y-12=0$

or $$5x+y-6=0 \quad \ldots(1)$$

From the second and third members,

$$x^2+y^2-6x-8y-1=0 \quad \ldots(2)$$

[To solve (1) and (2) for x and y.] Substitute the value of $y(=6-5x)$ from (1) in (2), and solve the resulting quadratic in x. Substitute these values of x one by one in $y=6-5x$, and find the corresponding values of y.]

3. Two points (0, 0), $(3, \sqrt{3})$ form with another point (x, y) an equilateral triangle. Find that point. [*P. U. 1951*]

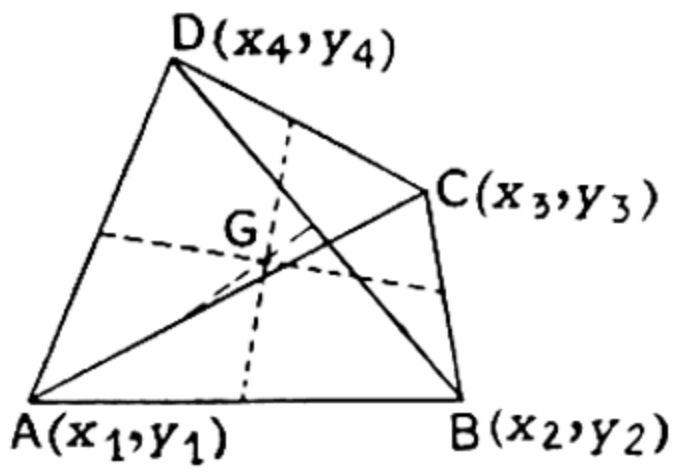
4. Find the point which is equidistant from (0, 0), (3, 1) and (6, 0).

5. Find the co-ordinates of the points of trisection of the line joining (3, −8) and (9, 4).

6. The line joining the points (−2, 4) and (4, −2) is divided into four equal parts ; find the co-ordinates of the points of section.

7. Prove that the lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another. [*P(P). U. 1951S*]

[**Sol.** Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ be the vertices of the quadrilateral. Then the mid-pt. of AB is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, and the mid-pt. of CD is $\left(\frac{x_3+x_4}{2}, \frac{y_3+y_4}{2}\right)$, ∴ the co-ordinates of G, the mid-pt. of the join of these mid-pts., are



$$x = \frac{\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}}{2} = \frac{x_1+x_2+x_3+x_4}{4},$$

$$y = \frac{\frac{y_1+y_2}{2} + \frac{y_3+y_4}{2}}{2} = \frac{y_1+y_2+y_3+y_4}{4}.$$

The *symmetry* of these results shows that the co-ordinates of the mid-pt. of the join of the mid-pts. of AD and BC, and the co-ordinates of the mid-pt. of the join of the mid-pts. of AC and BD are the same as those of G. ∴ the joins of the mid-pts. of AB and CD, AD and BC, AC and BD meet in G, and bisect one another.]

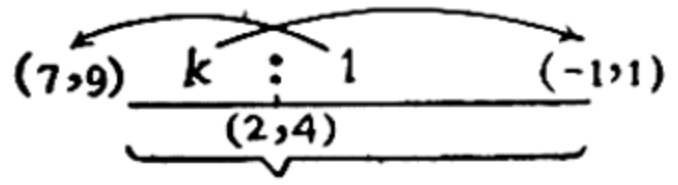
8. Length of a median. If (−1, 3), (1, −1) and (5, 1) are the vertices of a triangle, find the length of the median through the first vertex.

9. The vertices of a triangle are (2, 1), (5, 2), (3, 4). Find the co-ordinates of its centroid and the circumcentre. [*A. U. 1934*]

10. Show that the centroid of a triangle coincides with that of the triangle formed by joining the mid-points of its sides.

11. Converse problem. In what ratio does the point (2, 4) divide the join of (7, 9) and (−1, 1) ?

[**Hint.** Let the required ratio be $k : 1$.



The pt. which divides the join of (7, 9) and (−1, 1) in the ratio $k : 1$ is

$$\left[\frac{k(-1)+1(7)}{k+1}, \frac{k(1)+1(9)}{k+1}\right],$$

i.e., $\left(\frac{-k+7}{k+1}, \frac{k+9}{k+1}\right)$ ... (1)

∵ this is the same as (2, 4) ...(2)

∴ comparing (1) and (2), $\frac{-k+7}{k+1} = 2$ ... (3), $\frac{k+9}{k+1} = 4$ ... (4)

Find the value of k from (3) or (4).]

12. The mid-point of a line is (−4, −2), and one end of the line is (−6, 4). What are the co-ordinates of the other end ?

13. P, Q and R are three collinear points. P and Q are (3, 4) and (7, 7) respectively, and PR is equal to 10 units ; find the co-ordinates of R. [*P. U. 1946S*]

[**Hint.** It will be found that PQ=5, ∴ QR=10−5=5. ∴ Q is the mid-pt. of PR.]

14. (*a*) A, B, C are the points (−1, 5), (3, 1) and (5, 7) respectively. D, E, F are the middle points of BC, CA, AB respectively. Prove △ ABC=4△ DEF. [*P. U. 1945S*]

(*b*) Show analytically that the area of a triangle is four times the area of the triangle formed by joining the middle points of the sides. [*Pesh. U. 1955*]

15. Find the area of the triangle the co-ordinates of whose angular points are respectively $\left(at_1, \frac{a}{t_1}\right)$, $\left(at_2, \frac{a}{t_2}\right)$ and $\left(at_3, \frac{a}{t_3}\right)$.

16. A, B are the two points (3, 4) and (5,−2) ; find the point P such that PA=PB and △ PAB=10.

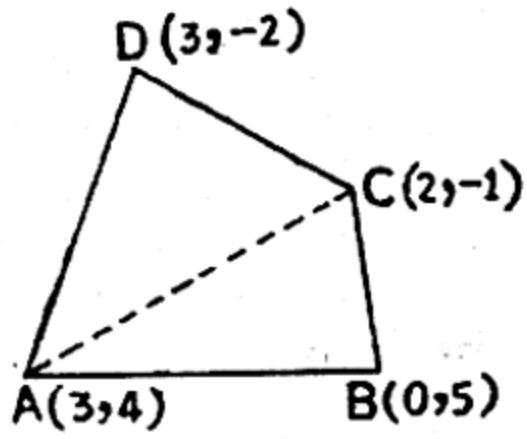
17. Prove that the three points (3a, 0), (0, 3b) and (a, 2b) are collinear. [*C. U.*]

****18. Area of a quadrilateral. Find the area of the quadrilateral whose vertices taken in order are :− (3, 4), (0, 5), (2, −1) and (3,−2).** [*J. & K. U. 1949*]

[**Short-cut.** The area of the quadrilateral
$= \frac{1}{2}$ [15−0+0−10−4−(−3)+12−(−6)]
[Rule (Note 1, Art. 9)]
$= \frac{1}{2}$ [22]=11.

½	3	4
	0 ×	5
	2 ×	−1
	3 ×	−2
	3 ×	4

Or thus : Hint. Let A(3, 4), B(0, 5), C(2,−1), D(3, −2) be the vertices of the quadrilateral. *Draw the diagonal AC (to divide the quadrilateral into triangles).*



Then quad. ABCD=△ ABC+△ ACD.]

****19. The co-ordinates of the angular points of a quadrilateral, taken in order, are (1, 1), (3, 4), (5, −2) and (4, −7) : find its area.** [*D. U. 1953*]

CHAPTER II

LOCUS AND ITS EQUATION

SECTION I

EQUATION OF A LOCUS

11. (*a*) **Locus. Def. The** *locus* **of a point moving under a given condition is the path traced out by the point.**

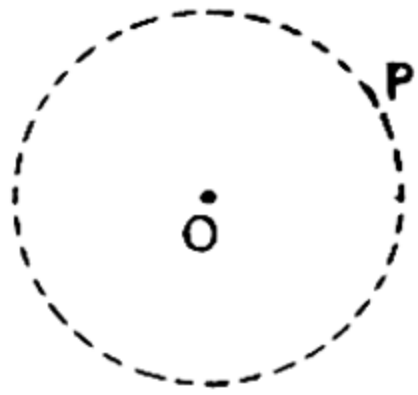


Illustration. The locus (path) of a point P, which moves so that its distance from a fixed point O is equal to $\frac{1}{2}$ inch, is a *circle*.

Thus *the locus of a point moving under a given condition is a definite* **curve**.

Note. '*Curve*' means 'a line *straight* or crooked.'

(*b*) **Equation of a locus. Def.** *The* **equation* of a locus (curve)** *is the equation in x and y which is satisfied by the co-ordinates of* **any** *point on the locus (curve), and by the co-ordinates of no other point.*

Cor. Important. If a point lies on a locus (curve), its co-ordinates satisfy the equation of the locus (curve) and conversely—

If the co-ordinates of a point satisfy the equation of a locus (curve), it lies on the locus (curve).

[**Rule to find whether a given point lies on a given locus (curve) :** *In the equation of the locus (curve) substitute the co-ordinates of the point for x, y. If the equation is satisfied, the point lies on the locus (curve), otherwise not.*]

An important abbreviation. *It is customary to shorten the expression "substitute the co-ordinates of the point for x, y" into the expression "substitute the co-ordinates of the point."*

EXAMPLES

1. Does the point (1, −1) lie on the locus whose equation is $3x-2y=5$? the point (2, −3) ?

[**Sol.** The equation of the locus is $3x-2y=5$ ... (1)

(*i*) In (1), substituting the co-ordinates of the pt. (1, −1), we get $3(1)-2(-1)=5$ or $5=5$, which is true.

$\therefore$ (1, −1) lies on the locus.

(*ii*) In (1), substituting the co-ordinates of the pt. (2, −3), we get $3(2)-2(-3)=5$ or $12=5$, which is *not* true.

$\therefore$ (2, −3) does *not* lie on the locus.]

2. Which of the points (2, 3), (3, −4), (−4, −3) lie on the locus whose equation is $x^2+y^2=25$?

---

*Or, more fully, the **Cartesian equation.**

12. 3-step Rule to find the equation of the locus of a point which moves under a given condition :

(*i*) *Let* (x, y) *be* **any** *point on the locus.*

(*ii*) *Write down the given condition.*

(*iii*) *Express this condition in co-ordinates, and simplify the result. This is the required equation.*

Example 1. *Find the equation of the locus of a point which moves so that its distance from the origin is always constant and equal to a.*

(*i*) Let (x, y) be *any* pt. on the locus.

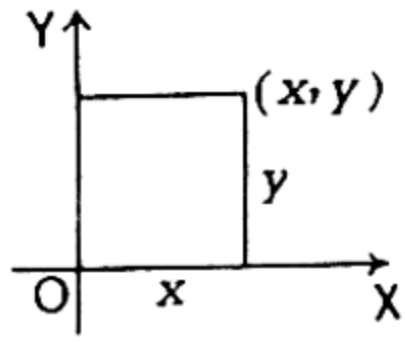
(*ii*) Then distance of (x, y) from the origin $(0, 0)=a$.

(*iii*) $\therefore \sqrt{(x-0)^2+(y-0)^2}=a$ or $\sqrt{x^2+y^2}=a$

or, squaring, $x^2+y^2=a^2$ or $x^2+y^2-a^2=0$, which is the required equation.

Important. *The equation of a locus must be written so that the powers of x and y are integral* (*i.e.*, whole numbers, not fractions). Note *squaring* in (*iii*).

Example 2. *A point moves so that its distance from the x-axis exceeds twice its distance from the y-axis by 3. Find the equation of its locus.*



(*i*) Let (x, y) be *any* pt. on the locus.

(*ii*) Then distance of (x, y) from the x-axis
= twice distance of (x, y) from the y-axis $+3$.

(*iii*) $\therefore \quad y=2x+3$

[$\because$ distance of (x, y) from the x-axis $=y$,
and distance from the y-axis = opposite side of rectangle
$=x$]

or $2x-y+3=0$, which is the required equation.

Note. Important. Distance of (x, y) from the x-axis = y, and distance of (x, y) from the y-axis = x.

EXAMPLES

1. Find the equation of the locus of a point which moves so that the square of its distance from $(0, 2)$ is equal to 4.

2. Find the equation of the locus of a point which moves so that it is always equidistant from the points $(1, -2)$ and $(-3, -4)$.

3. A point moves so that its distance from the point $(1, 0)$ is always twice its distance from the point $(0, -2)$. Find the equation to its locus.

4. Find the equation of the locus of a point which moves so that its distance from the point $(ak, 0)$ is k times its distance from the point $\left(\frac{a}{k}, 0\right)$.

**5. A point moves so that the sum of its distances from $(ae, 0)$ and $(-ae, 0)$ is $2a$; prove that its locus is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1.$$

[*P. U. 1944*]

[**Sol.** (*i*) Let (x, y) be *any* pt. on the locus.

(*ii*) Then distance of (x, y) from $(ae, 0)$
+ distance of (x, y) from $(-ae, 0)=2a$.

(*iii*) $\therefore \quad \sqrt{(x-ae)^2+y^2}+\sqrt{(x+ae)^2+y^2}=2a$ ...(1)

Now identically $[(x-ae)^2+y^2]-[(x+ae)^2+y^2]=-4aex$ ...(2)

Dividing (2) by (1),

$$\sqrt{(x-ae)^2+y^2}-\sqrt{(x+ae)^2+y^2}=\frac{-4aex}{2a}=-2ex \quad \text{...(3)}$$

Adding (1) and (3), $2\sqrt{(x-ae)^2+y^2}=2a-2ex$

or $\sqrt{(x-ae)^2+y^2}=a-ex.$

Squaring, $x^2-2aex+a^2e^2+y^2=a^2-2aex+e^2x^2$

or $x^2(1-e^2)+y^2=a^2(1-e^2)$

or, dividing by $a^2(1-e^2)$, $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$, which is the required (equation of the) locus.]

6. A point moves so that the difference of the squares of its distances from the two fixed points (a, 0) and (−a, 0) is constant=$2k^2$. Find the equation of its locus. [*C. U. 1944*]

[**Hint.** (*i*) Let (x, y) be *any* pt. on the locus.

(*ii*) Then square of distance of (x, y) from $(a, 0)$
−square of distance of (x, y) from $(-a, 0)=2k^2$ ...(1)

or square of distance of (x, y) from $(-a, 0)$
−square of distance of (x, y) from $(a, 0)=2k^2$ ...(2)

(*iii*) It will be found from (1) that the equation of the locus is $2ax=-k^2$ ...(3), and from (2), the equation is $2ax=k^2$ ...(4),

$\therefore$ combining (3) and (4), the equation of the locus is $2ax=\pm k^2$.]

[**Note.** "*Difference of A and B*" means "(1) A−B or (2) B−A."]

7. Find the equation of the locus of a point which moves so that its distance from the x-axis is m times its distance from the y-axis.

8. A point moves so that its distance from the axis of x is half its distance from the origin ; find the equation of its locus.

9. Find the equation of the locus of a point which moves so that its distance from the point $(a, 0)$ is equal to its distance from the y-axis.

10. A(1, 1), B(−2, 3) are two fixed points ; find the equation to the locus of a point P which moves so that the area of the triangle PAB=2.

SECTION II

LOCUS OF AN EQUATION

13. Locus of an equation. Def. The **locus of an equation** is the curve which passes through all the points whose co-ordinates satisfy the equation, and through no other points.

****14. Rule to trace the locus of an equation :**

(*i*) **Solve** *the equation for y in terms of* x [or for x in terms of y, whichever is simpler].

(*ii*) **Table of values.** *Find the values of y corresponding to low values of x* (such as 0, 1, 2, 3, ..., −1, −2, −3, ...) ; *form a table* (*of real corresponding values of x and y*).

(*iii*) *Choose a* **scale.**

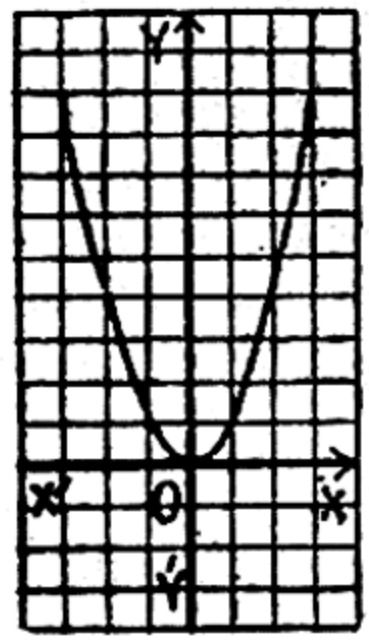
(*iv*) **Plot** *the points given by the above table, and draw a smooth curve through them.* This is the required locus.

Example. Trace the locus of the equation $y=x^2$.

(*i*) $y=x^2$. [Already solved for y in terms of x]

(*ii*) **Table of values.**

x	$y(=x^2)$	x	$y(=x^2)$
0	0		
1	1	−1	1
2	4	−2	4
3	9	−3	9
etc.	etc.	etc.	etc.



(*iii*) **Scale.** 1 small division=1.

(*iv*) Hence the locus shown above.

This curve is called a *parabola*.

CHAPTER III

THE STRAIGHT LINE

SECTION I

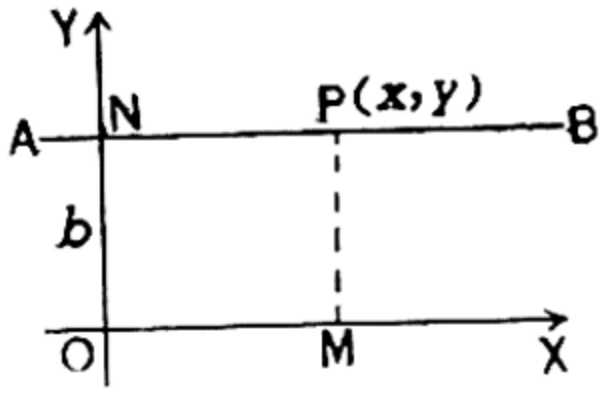
EQUATION OF A STRAIGHT LINE IN DIFFERENT FORMS

Three particular standard forms

I. A line parallel to the x-axis.

15. *To find the equation of a straight line parallel to the x-axis and at a distance b from it.*

Let AB be the line* ‖ to the x-axis, meeting the y-axis in N so that ON$=b$.



(*i*) Let P (x, y) be *any* pt. on the line. From P draw PM $\perp$ on the x-axis.

(*ii*) Then MP$=$ON$=b$.

(*iii*) $\therefore$ **y$=$b,** which is the required equation.

N.B. If the line is *above* the x-axis (as in the Fig.), b is positive, and if it is below, b is negative.

Cor. The equation of the x-axis is y$=$0.

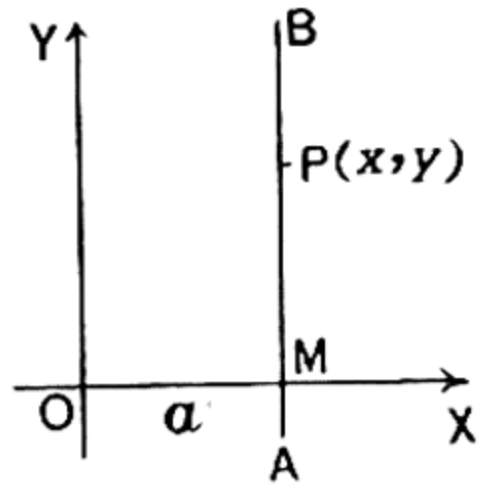
For, if the line coincides with the x-axis, then $b=0$.

***Note. Important.** *The word 'line' as used here and throughout the remainder of the book stands for* **'straight** *line'.*

II. A line parallel to the y-axis.

16. *To find the equation of a straight line parallel to the y-axis and at a distance a from it.*

Let AB be the line ‖ to the y-axis, meeting the x-axis in M so that OM$=a$.



(*i*) Let P(x, y) be *any* pt. on the line.

(*ii*) Then OM$=a$.

(*iii*) $\therefore$ **x$=$a,** which is the required equation.

N.B. If the line is to the *right* of the y-axis (as in the Fig.), a is positive, and if it is to the left, a is negative.

Cor. The equation of the y-axis is x$=$0.

For, if the line coincides with the y-axis, then $a=0$.

[**Aid to memory.** The equation of a line ‖ to the x-axis does not contain the x-co-ordinate so that its equation is $y=b$, and the equation of a line ‖ to the y-axis does not contain the y-co-ordinate so that its equation is $x=a$.

(Note further that x is associated with a and y with b.)

Similarly the equation of the x-axis is $y=0$ and that of the y-axis is $x=0$.]

Note. Arts. 15 and 16 show that the equation of a curve does not necessarily contain *both* x and y though generally it does.

EXAMPLES

1. Find the equation of a straight line parallel to the x-axis and at a distance (*i*) 1 unit above it ; (*ii*) 2 units below it.

[**Sol.** (*i*) Here $b=1$, $\therefore$ the equation of the line is $y=1$ $[y=b]$

or $y-1=0$.

(*ii*) Here $b=-2$, $\therefore$ the equation of the line is $y=-2$ $[y=b]$

or $y+2=0$.]

2. Find the equation of a straight line parallel to the y-axis and at a distance (*i*) 3 units to the right ; (*ii*) 4 units to the left.

3. *Find the equation of a straight line parallel to the x-axis and passing through the point* $(5, -6)$.

[**Sol.** (*i*) Let the required equation of the line be $y=b$ ...(1) [Art. 15]

(*ii*) $\because$ it passes thro' $(5, -6)$, these co-ordinates satisfy (1)

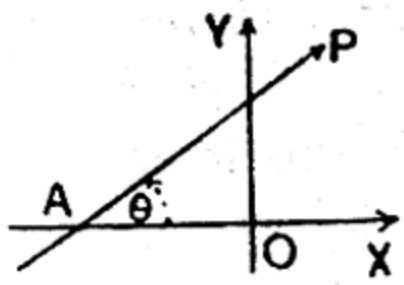
$\therefore -6=b$ or $b=-6$.

(*iii*) Substituting this value of b in (1), $y=-6$ or $y+6=0$, which is the required equation.]

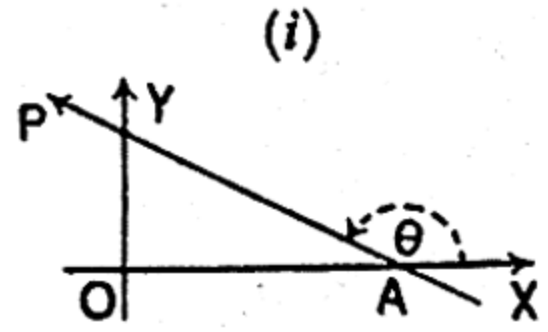
4. *Find the equation of a straight line parallel to the y-axis and passing through the point* $(-2, 3)$.

17. Slope of a line. Def. The slope (*or gradient*) of a line is the tangent of the angle which the part of the line above the x-axis makes with the positive direction of the x-axis.

[**Explanation.** *"The angle which the part of the line above the x-axis makes with the positive direction of the x-axis"* means "the angle measured positively (*i.e.*, in the counter-clockwise direction) from the positive direction of the x-axis to the part of the line above the x-axis." More fully, it means "the angle through which a revolving line starting from OX and revolving in the positive (*i.e.*, counter-clockwise) direction, would have to revolve in order to coincide with the part of the given line above the x-axis.]



(*i*)



(*ii*)

Thus if AP is the part of the line above the x-axis and $\angle XAP=\theta$, the slope of the line $=\tan\theta$.

Notation. The slope of a line is always denoted by m.

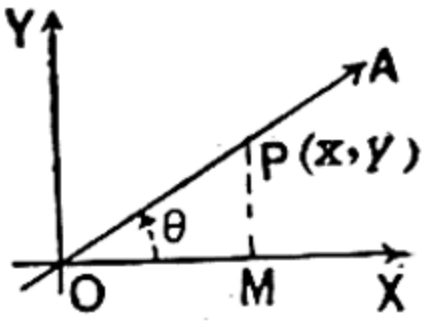
Thus $\mathbf{m=\tan\theta}$.

Note. θ is measured positively from $0°$ to $180°$.

III. A line through the origin.

18. *To find the equation of a straight line passing through the origin and making a given angle* θ *with the positive* direction of the x-axis.*

Let OA be the line, so that $\angle XOA=\theta$.



(*i*) Let P(x, y) be *any* pt. on the line. From P draw PM $\perp$ on the x-axis.

(*ii*) Then $\dfrac{MP}{OM}=\tan\theta$

$\therefore$ $MP=OM\tan\theta$.

(*iii*) $\therefore$ $y=x\tan\theta$,

or, putting $\tan\theta=m$, $\mathbf{y=mx}$, which is the required equation.

***An important abbreviation.** *The expression "with the positive direction of the x-axis" is shortened into the expression "with the x-axis".*

Example. *Find the equation of the straight line passing through the origin and inclined at 30° to the x-axis.*

Here $m=\tan 30°=\dfrac{1}{\sqrt{3}}$,

$\therefore$ the equation of the line is $y=\dfrac{1}{\sqrt{3}}.x$ $\quad [y=mx]$

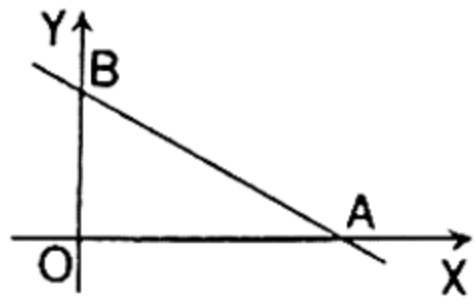
or $x-\sqrt{3}\ y=0$.

EXAMPLES

1. Find the equation of the straight line passing through the origin and making with the x-axis an angle of (*i*) 60° ; (*ii*) 90° ; (*iii*) $\dfrac{2\pi}{3}$; (*iv*) $\dfrac{5\pi}{6}$.

2. *Find the equations of the straight lines which pass through the origin and bisect the angles between the axes.*

19. (*a*) **Intercepts. Def.** Let a straight line AB meet the x-axis in A and the y-axis in B. Then (*i*) OA is called the **intercept of the line on the x-axis,**



(*ii*) OB is called the **intercept of the line on the y-axis,** and

(*iii*) the two together OA and OB (*in this particular order*) are called the **intercepts of the line on the axes.**

(*b*) **Rule for signs of intercepts.** (*i*) *The intercept on the x-axis is positive if it is measured to the right of the origin (as in the Fig. of Art. 19, (a)), and negative if measured to the left.*

(*ii*) *The intercept on the y-axis is positive if it is measured above the origin (as in the Fig. of Art. 19, (a)), and negative if measured below.*

Three general standard forms

I. Slope form or Tangent form.

20. To find the equation of a straight line which cuts off a given intercept c on the y-axis and is inclined* at a given angle θ to the x-axis.

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* **Note.** *" Is inclined at a given angle θ to the x-axis "* means "makes a given angle θ with the *positive direction* of the x-axis ".

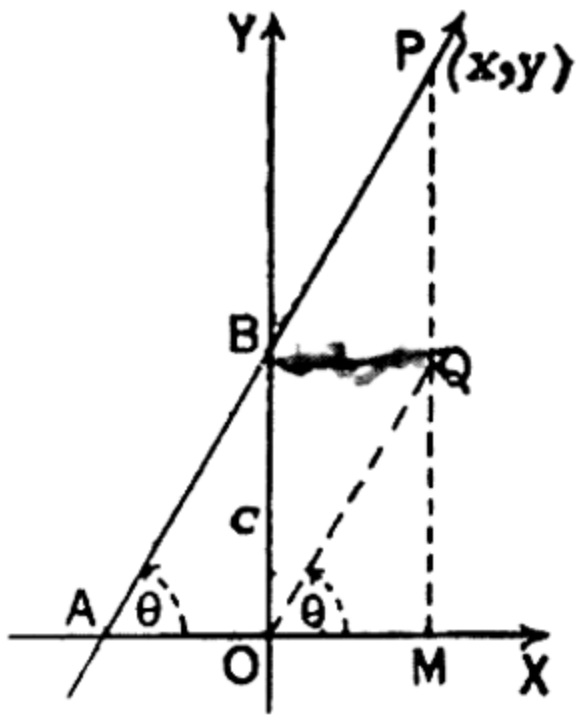
Let AB be the line meeting the axes in A, B, so that OB$=c$, $\angle XAB=\theta$.

(*i*) Let P(x, y) be *any* pt. on the line. From P draw PM $\perp$ on the x-axis and thro' O draw OQ $\parallel$ to AB to meet MP in Q.

Then $\angle MOQ=\angle XAB$ [Corresponding $\angle$s]
$=\theta$.

(*ii*) Now MP$=$MQ$+$QP.

[But $\frac{MQ}{OM}=\tan\theta$,
$\therefore$ MQ$=$OM tan θ, and QP$=$OB]
$=$OM tan $\theta+$OB.



(*iii*) $\therefore$ $y=x\tan\theta+c$,
or, putting tan $\theta=m$, $\mathbf{y=mx+c}$, which is the required equation.

N. B. If B is *above* O (as in the Fig.), c is positive, and if it is below, c is negative (Art. 19, (*b*), Rule for signs of intercepts).

Note 1. Slope form or Tangent form. Since in the equation $y=mx+c$, *m is the* **slope** *and c the* **intercept of the line** on the y-axis, this form of the equation of a line is called the **slope-intercept form** or simply the **slope form.**

Again, since m is the *tangent* of the angle which the part of the line above the x-axis makes with the positive direction of the x-axis, it is also called the **tangent form.**

****Note 2. Two particular cases of the equation $\mathbf{y=mx+c}$.**

Case I. If the line is $\parallel$ to the x-axis, then $m=\tan 0^\circ=0$, and the equation becomes $y=0.x+c$ or $y=c$. [*Cf.* Art. 15.]

Case II. If the line passes thro' the origin, then $c=0$, and the equation becomes $y=mx+0$ or $y=mx$. [*Cf.* Art. 18.]

****Cor.** *The equation of any straight line is of the first degree in x and y.*

Case I. If the line meets the y-axis, its equation is $y=mx+c$ (Art. 20).

Case II. If the line is $\perp$ to the y-axis, its equation is $x=a$ (Art. 16).

In either case the equation of the line is of the first degree in x and y.

The Converse of this Cor. is *important* and is proved in Art. 24.

Example. *Find the equation of a straight line making an angle of 135° with the x-axis and cutting the y-axis at a distance 2 below the origin.*

Here $m=\tan 135^\circ=\tan(180^\circ-45^\circ)=-\tan 45^\circ=-1$, $c=-2$,
$\therefore$ the equation of the line is $y=(-1)x+(-2)$ [$y=mx+c$]
or $x+y+2=0$.

EXAMPLES

1. (*a*) Obtain the equation of a straight line in the form $\mathbf{y=mx+c}$. [*Pesh. U. 1951*]

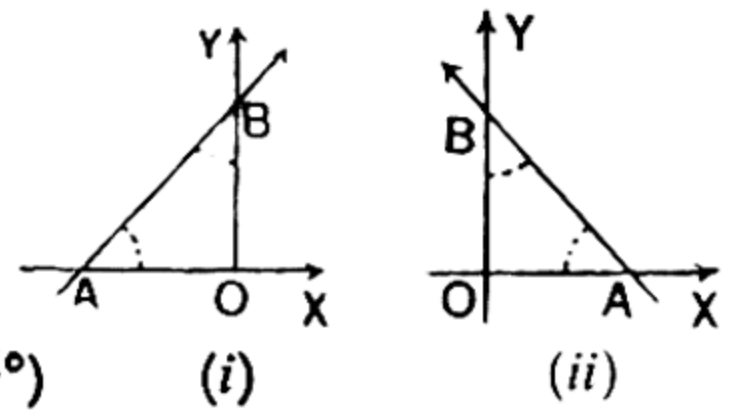
(*b*) Find the equation of a straight line (*i*) cutting off an intercept 2 from the positive direction of the axis of y and inclined at 45° to the x-axis. [*D. U. 1951*]

(*ii*) *cutting off an intercept 3 from the negative direction of the y-axis and inclined at 45° to OX.* [*C. U. 1939*]

2. Find the equation to the straight line cutting off an intercept -2 from the axis of y and inclined at an angle $\tan^{-1}\frac{3}{4}$ to the axis of x.

3. If a line is equally inclined to the axes, its slope $\mathbf{m}=\pm\mathbf{1}$.

[**Sol.** Let AB be the line meeting the axes in A, B, so that $\angle OAB = \angle OBA$, $\therefore$ each $=45°$.



Now in Fig. (*i*),

$m=\tan XAB=\tan 45°=1$ ... (1)

and in Fig. (*ii*),

$m=\tan XAB=\tan 135°=\tan(180°-45°)$

$=-\tan 45°=-1$ ... (2)

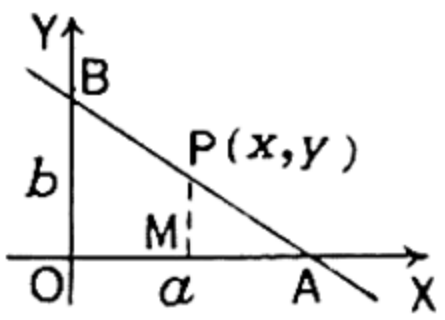
$\therefore$ combining (1) and (2), $\mathbf{m=\pm 1.}$]

*** 4.** *Find the equation of a straight line cutting off an intercept -3 from the y-axis and being equally inclined to the axes.*

II. Intercept form.

21. To find the equation of a straight line which cuts off given intercepts a and b from the axes.*

Let AB be the line meeting the axes in A, B, so that $OA=a$, $OB=b$.



(*i*) Let $P(x, y)$ be *any* pt. on the line. From P draw PM $\perp$ on the x-axis.

(*ii*) Then from the similar $\triangle$s PMA, BOA,

$$\frac{MP}{OB}=\frac{MA}{OA}=\frac{OA-OM}{OA}.$$

(*iii*) $\therefore$ $$\frac{y}{b}=\frac{a-x}{a}=1-\frac{x}{a}$$

or $$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1},$$ which is the required equation.

[**Aid to memory.** $$\frac{x}{\text{intercept on } x\text{-axis}}+\frac{y}{\text{intercept on } y\text{-axis}}=1.$$]

Note 1. Important. The line whose equation is $\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}$ meets the axes in the points (a, 0) and (0, b).

For, (see Fig. of Art. 21) $OA=a$, $\therefore$ A is $(a, 0)$;
and $OB=b$, $\therefore$ B is $(0, b)$.

Note 2. Intercept form. Since in the equation $\frac{x}{a}+\frac{y}{b}=1$, a and b are the *intercepts* of the line on the axes, this form of the equation of a line is called the **intercept form.**

****Note 3. Two particular cases of the equation $\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}$.**

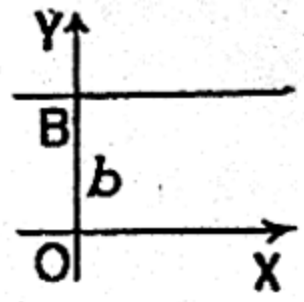
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*[*Pesh. U. 1954*]

Case I. If the line is $\parallel$ to the x-axis, then $a=\infty$ *,

and the equation becomes $\frac{x}{\infty}+\frac{y}{b}=1$ or $0+\frac{y}{b}=1$ or $y=b$.

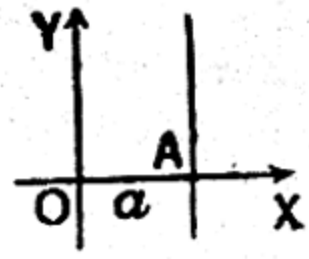
[*Cf.* Art. 15.]



Case II. If the line is $\parallel$ to the y-axis, then $b=\infty$ †,

and the equation becomes $\frac{x}{a}+\frac{y}{\infty}=1$ or $\frac{x}{a}+0=1$ or $x=a$.

[*Cf.* Art. 16.]



Example 1. *Find the equation of the straight line cutting off intercepts -2 and 3 from the axes.* [*C. U. 1944*]

Here $a=-2$, $b=3$, $\therefore$ the equation of the line is

$$\frac{x}{-2}+\frac{y}{3}=1 \qquad \left[\frac{x}{a}+\frac{y}{b}=1\right]$$

or $$-\frac{x}{2}+\frac{y}{3}=1 \text{ or } -3x+2y=6 \text{ or } 3x-2y+6=0.$$

Example 2. Find the equation of a line which makes equal intercepts on the axes and passes through the point (2, 3). [*P.U. 1948*]

Let the intercepts be a, a.

(i) Then the equation of the line is

$$\frac{x}{a}+\frac{y}{a}=1 \text{ or } x+y=a \quad ...(1)$$

(ii) $\because$ it passes thro' (2, 3), these co-ordinates satisfy (1)

$\therefore$ $2+3=a$ or $a=5$.

(iii) Substituting this value of a in (1), $x+y=5$ or $x+y-5=0$, which is the required equation.

EXAMPLES

1. (a) Obtain the equation of a straight line in the form

$$\frac{x}{a}+\frac{y}{b}=1.$$ [*D. U. 1953*]

1. (a) (Alternative form.) Find the equation of a straight line in terms of the intercepts which it makes on the axes. [*P. U.*]

(b) *Find the equation of the straight line cutting off intercepts 4 and 3 from the axes.* [*C.U.*]

2. (i) Find the equation of the straight line cutting off intercepts $-\frac{c}{m}$ and c from the axes.

(ii) Prove that the line whose intercepts on the axes of x and y are respectively -2 and 3 passes through the point (2, 6).

3. Find the equation of the line through the point (3, 4) which makes equal intercepts on the axes.

4. Find the equation of the straight line which passes through the point (5, 6) and has intercepts on the axes equal in magnitude but opposite in sign. [*C.U. 1934*]

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*Strictly speaking, $a\to\infty$ (read "a tends to infinity")

†Strictly speaking, $b\to\infty$.

5. *Find the lines through (2, −1) which cut the axes so that the intercepts are equal in magnitude.*

6. *A straight line passes through (2, 3) and the portion of the line intercepted between the axes is bisected at this point ; find its equation.* [*P. U.*]

[**Sol.** (*i*) Let the required equation of the line be $\frac{x}{a}+\frac{y}{b}=1$...(1)

(*ii*) Then it meets the axes in the pts. $(a, 0)$ and $(0, b)$. [Art. 21, Note 1]

The pt. which bisects the join of $(a, 0)$ and $(0, b)$ is $\left(\frac{a+0}{2}, \frac{0+b}{2}\right)$,

i.e., $\left(\frac{a}{2}, \frac{b}{2}\right)$ ... (2).

This is the same as (2, 3) ... (3)

$\therefore$ comparing (2) and (3), $\frac{a}{2}=2$, $\frac{b}{2}=3$, or $a=4$, $b=6$.

(*iii*) Substituting these values of a, b in (1), $\frac{x}{4}+\frac{y}{6}=1$

or $3x+2y=12$ or $3x+2y-12=0$, which is the required equation.]

7. Prove that the equation to the straight line which passes through the given point (x_1, y_1) and is such that the given point bisects the part intercepted between the axes is $\frac{x}{2x_1}+\frac{y}{2y_1}=1$.

[*D. U. 1949*]

8. *Find the equation to the straight line which passes through the point (−4, 1) and is such that the portion of it between the axes is divided by the point in the ratio 1 : 2.*

III. Perpendicular form.

22. To find the equation of a straight line in terms of the length of the perpendicular p from the origin upon it and the angle α which this perpendicular makes with the x-axis. *

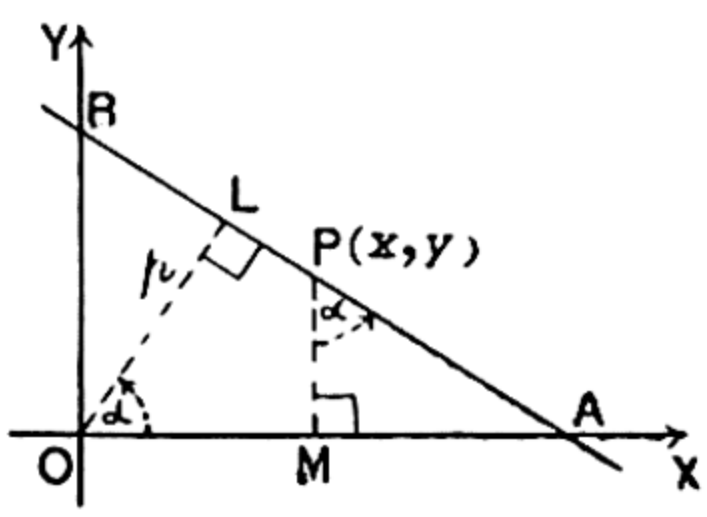
Let AB be the line meeting the axes in A, B, and OL the ⊥ from O upon it, so that $OL=p$, $\angle XOL=\alpha$.

(*i*) Let $P(x, y)$ be *any* pt. on the line. From P draw PM ⊥ on the *x*-axis.

(*ii*) Then $\angle MPA=90^\circ-\angle PAM$ [From the rt. ∠d △ PMA]

$=\angle AOL$ [From the rt. ∠d △ OLA]

$=\alpha$.



Now $\frac{OL}{OA}=\cos\alpha$ [From the △ OLA]

$\therefore$ $OL=OA\cos\alpha=(OM+MA)\cos\alpha=OM\cos\alpha+MA\cos\alpha$

$\left[\text{But } \frac{MA}{MP}=\tan\alpha, \quad \therefore MA=MP\tan\alpha\right]$

$\therefore$ $OL=OM\cos\alpha+MP\tan\alpha\cos\alpha$

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*[*P (P). U. 1956 S*]

$$=OM\cos\alpha+MP\frac{\sin\alpha}{\cos\alpha}\ .\ \cos\alpha. \qquad [\text{ Cancel } \cos\alpha\ ]$$

(*iii*) $\therefore\ p=x\cos\alpha+y\sin\alpha$

or **$x\cos\alpha+y\sin\alpha=p$,** which is the required equation.

Note 1. Perpendicular form. Since in the equation $x\cos\alpha+y\sin\alpha=p$, p is the length of the *perpendicular* from the origin upon the line, this form of the equation of a line is called the **perpendicular form.**

Note 2. Important. p is always positive, and α is measured positively from 0° to 360°.

EXAMPLES

1. Obtain the equation of a straight line in the form

$x\cos\alpha+y\sin\alpha=p$. [*D. U. 1955*]

2. *The length of the perpendicular from the origin on a straight line is 3 and the perpendicular is inclined to the x-axis at 120°. Find the equation.*

[**Sol.** Here $p=3,\ \alpha=120^\circ,$ $\therefore$ the equation of the line is

$x\cos 120^\circ+y\sin 120^\circ=3$ [$x\cos\alpha+y\sin\alpha=p$]

[But $\cos 120^\circ=\cos(180^\circ-60^\circ)=-\cos 60^\circ=-\frac{1}{2}$,

$\sin 120^\circ=\sin(180^\circ-60^\circ)=\sin 60^\circ=\frac{\sqrt{3}}{2}$]

or $x\left(-\frac{1}{2}\right)+y\left(\frac{\sqrt{3}}{2}\right)=3$ or $x-\sqrt{3}y+6=0$.]

3. Find the equation to the straight line on which the perpendicular from the origin is of length 2 and the inclination of this perpendicular to the x-axis is (*i*) 150° ; (*ii*) $\frac{5\pi}{3}$.

4. State the geometrical meanings of the constants in

(*i*) $\mathbf{y=mx+c}$, (*ii*) $\mathbf{\frac{x}{a}+\frac{y}{b}=1}$, **and** (*iii*) $\mathbf{x\cos\alpha+y\sin\alpha=p}$.

[*Pesh. U. 1955*]

23. Note on the Rule of cross-multiplication to solve the equations and

$a_1x+b_1y+c_1z=0$,

$a_2x+b_2y+c_2z=0$ simultaneously.

The equations are $a_1x+b_1y+c_1z=0$ ... (1)

$a_2x+b_2y+c_2z=0$ ... (2)

$\therefore$ solving (1) and (2) by cross-multiplication,

$$\begin{array}{ccccc} x & & y & & z \\ \big\lceil & & \big\lceil & & \big\lceil \\ b_1 & \times & c_1 & \times & a_1 & \times & b_1 \\ b_2 & & c_2 & & a_2 & & b_2 \end{array} \qquad \ldots (3)$$

or $$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{z}{a_1b_2-a_2b_1}* \qquad \ldots(4)$$

[**Explanation.** *Omit the coefficients of x from the given equations (1) and (2), and write in two rows the coefficients of y, z and x in order, repeating at the end the column at the head and draw diagonals thus getting:*

---

*This is not a *proof* of the Rule, but only a quick method of writing down the *answer*.

$$\begin{matrix} b_1 & & c_1 & & a_1 & & b_1 \\ & \times & & \times & & \times & \\ b_2 & & c_2 & & a_2 & & b_2 \end{matrix}$$

In the first row connect each coefficient with the next by a horizontal line ⌐¬, *and write x, y, z above these lines thus getting :*

$$\overbrace{\begin{matrix} b_1 & & c_1 & & a_1 & & b_1 \\ & \times & & \times & & \times & \\ b_2 & & c_2 & & a_2 & & b_2 \end{matrix}}^{x \qquad\quad y \qquad\quad z} \quad \ldots(3)$$

Simplification. *To obtain the denominator of x,* take the products of the columns below x in (3) crossways* putting the minus sign between them, thus getting : $\begin{matrix} b_1 \\ \times \\ b_2 \end{matrix} \begin{matrix} c_1 \\ \\ c_2 \end{matrix} = b_1c_2 - b_2c_1$. Similarly for the denominators of y and z. Thus getting :

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \quad \ldots(4)]$$

24. To prove that any equation of the first degree in x and y represents a straight line.†

[Method of area.] Let the equation of the first degree in x and y be $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$‡ ...(1)

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be *any* three pts. on the locus of the equation (1). Then the co-ordinates of these pts. satisfy (1)

$\therefore$ $\mathrm{A}x_1 + \mathrm{B}y_1 + \mathrm{C} = 0$ ...(2)

$\mathrm{A}x_2 + \mathrm{B}y_2 + \mathrm{C} = 0$ ...(3)

$\mathrm{A}x_3 + \mathrm{B}y_3 + \mathrm{C} = 0$ ...(4)

Solving (2) and (3) for A, B, C by cross-multiplication,

$$\overbrace{\begin{matrix} y_1 & & 1 & & x_1 & & y_1 \\ & \times & & \times & & \times & \\ y_2 & & 1 & & x_2 & & y_2 \end{matrix}}^{\mathrm{A} \qquad\quad \mathrm{B} \qquad\quad \mathrm{C}}$$

[Art. 23, Rule]

$$\therefore \quad \frac{\mathrm{A}}{y_1 - y_2} = \frac{\mathrm{B}}{x_2 - x_1} = \frac{\mathrm{C}}{x_1y_2 - x_2y_1} = k \text{ (say)}$$

$\therefore$ $\mathrm{A} = k(y_1 - y_2)$, $\mathrm{B} = k(x_2 - x_1)$, $\mathrm{C} = k(x_1y_2 - x_2y_1)$.

Substituting these values of A, B, C in (4),

$k(y_1 - y_2)x_3 + k(x_2 - x_1)y_3 + k(x_1y_2 - x_2y_1) = 0$ [Cancel k]

or $x_3\,y_1 - x_3\,y_2 + x_2\,y_3 - x_1\,y_3 + x_1\,y_2 - x_2\,y_1 = 0$

or, dividing by 2,

$\frac{1}{2}[x_1\,y_2 - x_2\,y_1 + x_2\,y_3 - x_3\,y_2 + x_3\,y_1 - x_1\,y_3] = 0$,

i.e., the area of the triangle formed by the three pts. is zero.

$\therefore$ the three pts. lie in a *st. line.*

But these are *any* three pts. on the locus.

$\therefore$ the locus is a st. line.

Note 1. Important. General form. Since the equation $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ can, by a proper choice of A, B, C, be made to represent *any* straight line, this form of the equation of a line is called the **general form.** Notice its

---

*Hence the name "the Rule of *cross*-multiplication." †[*P. U. 1956*]

‡Called the *general* equation of the first degree, because it contains *all* possible terms in x and y of the *first* and lower degree.

Characteristic. *R. H. S. is zero.*

Note 2. The equation of a straight line contains two constants.

For, we have the following (general) standard forms :

Name	Equation	Constants
Slope form or Tangent form	$y=mx+c.$	m and $c.$
Intercept form	$\frac{x}{a}+\frac{y}{b}=1.$	a and $b.$
Perpendicular form	$x\cos\alpha+y\sin\alpha=p.$	p and $\alpha.$

The general form $Ax+By+C=0$, of the equation of a straight line apparently contains three constants A, B, C, but in reality it contains only two.

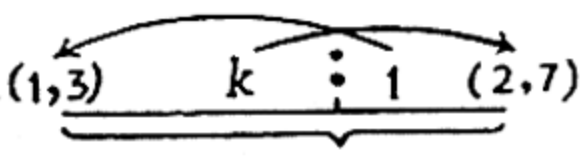
For, *solving the equation for y*, $By=-Ax-C$,

or $y=-\frac{A}{B}x-\frac{C}{B}$; or, putting $-\frac{A}{B}=m$, $-\frac{C}{B}=c$, $y=mx+c$, which contains only two constants m and c.

****Note 3.** *A straight line is determined by two conditions.* For, the equation of a straight line contains two constants which are determined from the two equations obtained by using the two given conditions.

Example. In what ratio is the line joining (1, 3) and (2, 7) divided by $3x+y=9$? [*P. U. 1942*]

Let the required ratio be $k : 1$. The pt. which divides the join of (1, 3) and (2, 7) in the ratio $k : 1$ is



$\left[\frac{k(2)+1(1)}{k+1}, \frac{k(7)+1(3)}{k+1}\right]$, *i.e.*, $\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1}\right)$.

$\because$ this pt. lies on $3x+y=9$, $\therefore 3\left(\frac{2k+1}{k+1}\right)+\left(\frac{7k+3}{k+1}\right)=9$

or $6k+3+7k+3=9k+9$ or $4k=3$ or $k=\frac{3}{4}$, $\therefore$ the required ratio is 3 : 4.

EXAMPLES

1. Prove that the equation $lx+my-1=0$ always represents a straight line. [*P. U. 1947*]

2. Prove that the middle point of the join of (3, 4) and (5, 1) is on the line $x-2y+1=0$. [*D. U. 1944*]

3. *A point moves so that the sum of its distances from two perpendicular lines is constant* $(=a)$. *Show that the locus is a straight line.*

4. *Show that the locus of a point which is equidistant from two points is a straight line.* [*P. U.*]

5. **Determine the ratio of the segments in which the straight line joining (2, 3) to (−1, 4) is cut by the straight line $x+y+1=0$.** [*P. U. 1932S*]

6. **Find in what ratios the join of (−2, 2) and (4, 5) is cut by the axes of co-ordinates.** [*P(P). U.*]

[**Hint.** (*i*) The equation of the x-axis is $y=0$; (*ii*) the equation of the y-axis is $x=0$.]

7. *Prove that the line* $y-x+2=0$ *bisects the join of* (2, 1) *and* (−2, −5).

[Short-cut.** The pt. which bisects the join of (2, 1) and (−2, −5) is $\left[\frac{2+(-2)}{2}, \frac{1+(-5)}{2}\right]$, *i.e.*, (0, −2).

Substituting these co-ordinates in the equation of the line $y-x+2=0$, we get $-2-0+2=0$ or $0=0$, which is true

$\therefore$ (0, −2) lies on the line.]

8. *Prove that the line $y-x+2=0$ cuts the line joining (3, −1) and (8, 9) in the ratio 2 : 3.* [*D. U. 1946*]

Reduction of the general equation to standard forms

I. Reduction to the tangent form.

25. *To reduce the equation $Ax+By+C=0$ to the form $y=mx+c$.*

[In the required form $y=mx+c$, (*i*) *only y* occurs on the L.H.S. and (*ii*) its coefficient = 1.]

The equation is $Ax+By+C=0$.

(*i*) Transposing, $By=-Ax-C$. [Only *y*-term on L.H.S.]

(*ii*) Dividing thro' out by B, $y=-\frac{A}{B}x-\frac{C}{B}$,

which is the required form.

Note. Comparing this with $y=mx+c$,

$m=-\frac{A}{B}$, $c=-\frac{C}{B}$, *i.e.*, slope $=-\frac{A}{B}$.

[**Rule to reduce the general equation (of a straight line) to the tangent form :**

(*i*) *Keep only the y-term on the L. H. S., and transpose all others to the R. H. S.. and* (*ii*) *divide throughout by the coefficient of y.*

In short : *Solve the given equation for y in terms of x.*]

Cor. Important. Rule to write down the slope of a line whose equation is given :

$$\textbf{slope} = -\frac{\textbf{coefficient of x}}{\textbf{coefficient of y}}$$ **in the L.H. S. of the equation (R. H. S. being zero).** [See Art. 25, Note.]

EXAMPLES

1. (*i*) *Reduce the equation $2x-3y+6=0$ to the tangent form.*

(*ii*) *Write the equation $x+2y-3=0$ in the form $y=mx+c$.*

2. What is the inclination to the *x*-axis, of the line $\sqrt{3}x-y+2=0$?

[**Sol.** Slope $=-\frac{\sqrt{3}}{-1}$ $\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y}\text{ (Art. 25, Cor.)}\right]$

$=\sqrt{3}$

$\therefore$ $\tan\theta=\sqrt{3}=\tan 60°$, $\therefore$ $\theta=60°$, which is the required angle.]

3. (*a*) Find the slope of the line $x\cos\alpha+y\sin\alpha=p$. [*D.U. 1955*]

(*b*) Find the inclination to the *x*-axis, of the lines

(*i*) $x+y=1$; (*ii*) $x+\sqrt{3}y-2\sqrt{3}=0$; (*iii*) $x-2y+1=0$.

4. Find the slope of a line which cuts off from the axes (*i*) equal intercepts ; (*ii*) intercepts equal in magnitude but opposite in sign.

II. Reduction to the intercept form.

26. To reduce the equation $Ax+By+C=0$ to the form $\frac{x}{a}+\frac{y}{b}=1$. *

[In the required form $\frac{x}{a}+\frac{y}{b}=1$, (*i*) the constant term occurs on the R. H. S. and is=1, and (*ii*) the coefficients of x and y occur as their denominators.]

The equation is $Ax+By+C=0$.

(*i*) Transposing, $Ax+By=-C$. [Constant term on R. H. S.]

Dividing thro'out by $-C$, $\frac{A}{-C}x+\frac{B}{-C}y=1$ [Constant term=1]

(*ii*) or $\frac{x}{\frac{-C}{A}}+\frac{y}{\frac{-C}{B}}=1$, [Coeffs. of x and y as their denoms.]

which is of the required form.

Note. Comparing this with $\frac{x}{a}+\frac{y}{b}=1$, $a=-\frac{C}{A}$, $b=-\frac{C}{B}$,

i.e., intercept on the x-axis $=-\frac{C}{A}$, intercept on the y-axis $=-\frac{C}{B}$.

[Rule to reduce the general equation (of a straight line) to the intercept form :

(*i*) *Transpose the constant term to the R. H. S., and divide throughout by it, and* (*ii*) *write the coefficients of x and y as their denominators.*]

Cor. Important. Rule to find the intercepts of a line on the axes :

In the equation of the line, put $y=0$, and find the value of x ; this is the intercept on the x-axis. Again in the equation of the line, put $x=0$, and find the value of y ; this is the intercept on the y-axis.

[**Proof.** Putting $y=0$ in $Ax+By+C=0$ ... (1), $Ax+C=0$,

$\therefore \quad x=-\frac{C}{A}$, which is the intercept on the x-axis.

Again putting $x=0$ in (1), $By+C=0$, $\therefore \quad y=-\frac{C}{B}$, which is the intercept on the y-axis.]

[**Aid to memory.** To find the intercept of a line on the x-axis, use the *equation of the x-axis, i.e.,* $y=0$. Similarly to find the intercept on the y-axis.]

EXAMPLES

1. Reduce the equation $3x-4y+5=0$ to the intercept form.
2. *Find the intercepts on the axes made by the line*
   (*i*) $4x=5y-20$; (*ii*) $x\cos\alpha+y\sin\alpha=\sin 2\alpha$.

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* [*D. U. 1951*]

III. Reduction to the perpendicular form.

27. To reduce the equation $Ax+By+C=0$ to the form $x\cos\alpha+y\sin\alpha=p$.

[In the required form $x\cos\alpha+y\sin\alpha=p$,
(*i*) (coeff. of x)^2+(coeff. of y)^2=$\cos^2\alpha+\sin^2\alpha=1$, and (*ii*) the constant term occurs on the R. H. S. and is positive.]

The equation is $Ax+By+C=0$.

Multiplying thro' out by k, $kAx+kBy+kC=0$ ... (1)

(*i*) Putting $(kA)^2+(kB)^2=1$, [(coeff. of x)^2+(coeff. of y)^2=1]

$k^2(A^2+B^2)=1$, $\therefore\quad k=\dfrac{1}{\sqrt{A^2+B^2}}$.

[Taking +ve sign with the square root]

Substituting this value of k in (1),

$$\frac{A}{\sqrt{A^2+B^2}}.x+\frac{B}{\sqrt{A^2+B^2}}.y+\frac{C}{\sqrt{A^2+B^2}}=0.$$

(*ii*) Transposing,

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}} \quad \text{... (2)}$$

[Constant term on R. H. S.]

Case I. If C is +ve, the R. H. S. of (2) is −ve,

$\therefore$ changing the signs thro' out,

[To make constant term on R. H. S. +ve]

$$-\frac{A}{\sqrt{A^2+B^2}}x-\frac{B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}},$$

which is of the required form.

Note. Comparing this with $x\cos\alpha+y\sin\alpha=p$,

$$\cos\alpha=-\frac{A}{\sqrt{A^2+B^2}},\ \sin\alpha=-\frac{B}{\sqrt{A^2+B^2}},\ p=\frac{C}{\sqrt{A^2+B^2}}.$$

Case II. If C is −ve, the R. H. S. of (2) is +ve,

$\therefore$ (2) is of the required form.

Note. Comparing (2) with $x\cos\alpha+y\sin\alpha=p$,

$$\cos\alpha=\frac{A}{\sqrt{A^2+B^2}},\ \sin\alpha=\frac{B}{\sqrt{A^2+B^2}},\ p=-\frac{C}{\sqrt{A^2+B^2}}.$$

[**Rule to reduce the general equation of a straight line ($Ax+By+C=0$) to the perpendicular form :**

(*i*) *Divide thro' out by* $\sqrt{A^2+B^2}$, *i.e., by* $\sqrt{(coeff.\ of\ x)^2+(coeff.\ of\ y)^2}$, *and*

(*ii*) *transpose the constant term to the R. H. S., and make it positive (by changing the signs throughout, if necessary).*]

Example. Reduce the equation

(a) $\sqrt{3}x-y+2=0$; (b) $3x+4y-15=0$ to the perpendicular form.

(*a*) The equation is $\sqrt{3}x-y+2=0$.

(*i*) Dividing thro' out by $\sqrt{(\sqrt{3})^2+(-1)^2}$

[$\sqrt{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2}$]

i.e., by 2, $\frac{\sqrt{3}}{2}x-\frac{1}{2}y+1=0.$

(*ii*) Transposing, $\frac{\sqrt{3}}{2}x-\frac{1}{2}y=-1$ [Constant term on R.H.S.]

or, changing the signs thro' out, [To make constant term on R.H.S. +ve]

$$-\frac{\sqrt{3}}{2}x+\frac{1}{2}y=1 \quad \ldots (1),$$

which is of the required form.

Note. Comparing this with $x\cos\alpha+y\sin\alpha=p$,

$\cos\alpha=-\frac{\sqrt{3}}{2}$, $\sin\alpha=\frac{1}{2}$, $p=1$.

**Consider the values of* α *between* $0°$ *and* $360°$.

$\cos\alpha=-\frac{\sqrt{3}}{2}=-\cos 30°=\cos(180°-30°)$ or $\cos(180°+30°)$

$=\cos 150°$ or $\cos 210°$

$\therefore \quad \alpha=150°$ or $210°$ ... (2)

$\sin\alpha=\frac{1}{2}=\sin 30°$ or $\sin(180°-30°)=\sin 30°$ or $\sin 150°$

$\therefore \quad \alpha=30°$ or $150°$ ... (3)

From (2) and (3), the common value is $\alpha=150°$.

$\therefore$ equation (1) may further be written as

$x\cos 150°+y\sin 150°=1$ ...(4),

which is of the required form.

Important. *When the value of* α *can be found with the help of the t-ratios of standard angles* ($0°$, $30°$, $45°$, $60°$, $90°$) *as above, the required equation* (*1*) *should further be written as* (*4*).

(*b*) The equation is $3x+4y-15=0$.

(*i*) Dividing thro'out by $\sqrt{(3)^2+(4)^2}$ [$\sqrt{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2}$]

i.e., by 5, $\frac{3}{5}x+\frac{4}{5}y-3=0.$

(*ii*) Transposing,

$\frac{3}{5}x+\frac{4}{5}y=3$, [Constant term on R. H. S. which becomes +ve]

which is of the required form.

Note. Comparing this with $x\cos\alpha+y\sin\alpha=p$,

$\cos\alpha=\frac{3}{5}$, $\sin\alpha=\frac{4}{5}$, $p=3$.

EXAMPLES

1. Reduce the following equations to the perpendicular form

(*i*) $x+\sqrt{3}y+4=0$. (*ii*) $5x-12y-13=0$.

(*iii*) $x+1=0$. (*iv*) $2y-3=0$.

2. Reduce the equation $x+y+5=0$ to the form $x\cos\alpha+y\sin\alpha=p$ and find the values of p and α. [*P. U. 1943 S*]

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* Compare this with steps (*i*) and (*ii*) in the Rule in the author's *New Intermediate Plane Trigonometry* (*6th Edition*), Art. 80.

3. The equation of a straight line may be put in the form $x\cos\theta+y\sin\theta=p$; determine the values of θ and p of the line $x+\sqrt{3}y=4$. [*P. U. 1945S*]

4. *Reduce the standard forms* ($y=mx+c$ *and* $\frac{x}{a}+\frac{y}{b}=1$) *to the form* $x\cos\alpha+y\sin\alpha-p=0$ *and indicate the length of the perpendicular from the origin upon each line.*

Three point-forms of the equation of a line

I. Point-slope form.

28. *To find the equation of a straight line drawn through a given point* (x_1, y_1) *in a given direction, i.e., making a given angle* θ *with the x-axis.*

The slope of the line $=\tan\theta=m$ (say).

(*i*) The equation of *any* line whose slope is m, is $y=mx+c$ ... (1) [Art. 20]

(*ii*) If it passes thro' (x_1, y_1), then $y_1=mx_1+c$ ... (2)

(*iii*) Subtracting (2) from (1), $\mathbf{y-y_1=m(x-x_1)}$, which is the required equation.

Note 1. Point-slope form. Since in the equation $y-y_1=m(x-x_1)$, (x_1, y_1) is the *point* through which the line passes and m the *slope* of the line, this form of the equation of a line is called the **point-slope form**.

****Note 2. A particular case of the equation $\mathbf{y-y_1=m(x-x_1)}$.**

If the line passes through the origin, then $x_1=0$, $y_1=0$, and the equation becomes $y-0=m(x-0)$ or $y=mx$. [*Cf.* Art. 18.]

Cor. Important. The equation of any line through a given point $(\mathbf{x_1, y_1})$ is $\mathbf{y-y_1=m(x-x_1)}$.

Note. The constant m is determined from the second condition satisfied by the line.

EXAMPLES

1. *Find the equation of the line through the point* $(1, -2)$ *making an angle of 135° with the x-axis.* [*D. U.*]

[**Sol.** Here $m=\tan 135°=-1$, $\therefore$ the equation of the line thro' $(1, -2)$ is $y-(-2)=-1\,(x-1)$ [$y-y_1=m(x-x_1)$]

or $y+2=-x+1$

or $x+y+1=0$.]

2. *Find the equation of the line through* $(1, 2)$ *which makes an angle* $\tan^{-1}3$ *with the axis of x.*

3. Find the equation of a straight line which makes an angle of $90°+\alpha$ with the axis of x, and which passes through the point ($p\cos\alpha$, $p\sin\alpha$). [*P. U.*]

4. Prove that the straight line which makes an angle $\tan^{-1}2$ with the x-axis, and which cuts the y-axis in the point $(0, -2)$, passes through the point $(1, 0)$.

****[Another method. Hint.** Here $\theta=\tan^{-1}2$, $\therefore$ $m=\tan\theta=2$, $c=-2$, Use the slope form ($y=mx+c$).]

II. Two-point form.

29. To find the equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2)*.

(*i*) The equation of *any* line passing thro' (x_1, y_1) is
$y-y_1=m(x-x_1)$ ...(1) [Art. 28, Cor.]

(*ii*) If it passes thro' (x_2, y_2), then
$y_2-y_1=m(x_2-x_1)$

$$\therefore \qquad m=\frac{y_2-y_1}{x_2-x_1} \quad ...(2)$$

(*iii*) Substituting this value of m in (1),

$$\mathbf{y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)},$$

which is the required equation.

Cor. Important. Rule to write down the slope of the line passing thro' the two points (x_1, y_1) and (x_2, y_2) :

$$\mathbf{slope=\frac{y_2-y_1}{x_2-x_1}}.$$

Note. Two-point form. Since in the equation $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$, (x_1, y_1) and (x_2, y_2) are the *two points* through which the line passes, this form of the equation of a line is called the **two-point form.**

[**Aid to memory.** The slope of the line passing thro' (x_1, y_1) and (x_2, y_2),

$$=\frac{y_2-y_1}{x_2-x_1}.$$

$\therefore$ the equation of the line passing thro' (x_1, y_1) and having slope $\frac{y_2-y_1}{x_2-x_1}$ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1). \qquad [y-y_1=m(x-x_1)]$$

Example 1. Find the equation of the line joining the points (1, 1), (2, 3). [*B.U.*]

The equation of the line is

(1)	(2)
(1, 1)	(2, 3)

$$y-1=\frac{3-1}{2-1}(x-1) \qquad \left[y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\right]$$

$$=2(x-1)=2x-2$$

or $\quad 2x-y-1=0.$

[**Check.** This equation is satisfied by the co-ordinates of the two given pts. (1, 1), (2, 3), thus $2-1-1=0$ or $0=0$, $2(2)-3-1=0$ or $0=0$.]

Example 2. Show that the three points (1, 4), (3, −2) and (−3, 16) are collinear. [*P. U.*]

[Another method.]

(*i*) The equation of the line passing thro' (1, 4) and (3, −2) is

(1)	(2)
(1, 4)	(3, −2)

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*[*Bar. U. 1954*]

$$y-4=\frac{-2-4}{3-1}(x-1) \qquad \left[y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\right]$$

$$=-3(x-1)=-3x+3$$

or $3x+y-7=0.$

(*ii*) Substituting in this equation the co-ordinates of $(-3, 16)$, we get

$3(-3)+16-7=0$ or $0=0$, which is true.

$\therefore$ the third pt. lies on the line passing thro' the first two, *i.e.*, the three pts. are collinear.

For another method see Art. 10, solved Ex.

Note. The above method enables us not only to prove that the three points are collinear, but also to find the *equation of the line* on which they lie.

[**Another method to prove that three given points are collinear :** (*i*) *Find the equation of the line passing through two points.* (*ii*) *Substitute in this equation the co-ordinates of the third point, and show that it is satisfied by them.*]

EXAMPLES

1. (*a*) **Find the equation of a straight line which passes through two given points.** [*D. U. 1948*]

(*b*) *Find the equations to the sides of the triangle the co-ordinates of whose angular points are* $(1, 4)$, $(2, -3)$, *and* $(-1, -2)$. [*P. U. 1944S*]

2. Find the equation of the line

(*i*) **joining the origin to the point** (x_1, y_1) ;

(*ii*) passing through the points (a, b) and (b, a) ;

(*iii*) passing through the points $\left(at_1, \frac{a}{t_1}\right)$ and $\left(at_2, \frac{a}{t_2}\right)$.

3. Equation of the median. (*i*) *Given the triangle* $A(10,4)$, $B(-4,9)$, $C(-2, -1)$, *find the equation of the median through* A. [*D. U. 1942*]

(*ii*) *The vertices of a triangle are* $(1, 0)$, $(0, 1)$ *and* $(2, 3)$. *Show that the median through the first vertex is given by* $x-1=0$. [*B.U. 1946*]

4. Show that the points (a, 0), (0, b), (3a, −2b) are on a straight line. Find its equation. [*P. U. 1936*]

5. Show that the points whose co-ordinates are $(5, 1)$, $(1, -1)$ and $(11, 4)$ lie on a straight line, and find its intercepts on the axes.

[*D. U. 1953*]

6. Find the slope of the line joining $(3, 0)$ and $(2, \sqrt{3})$. What is its inclination to the x-axis ?

7. In what ratio is the line joining (2, 1) and (5, 2) divided by the line joining (3, 2) and (5, 0) ? [*P. U. 1943*]

[**Hint.** Find the equation of the line passing thro' the second pair of pts. and proceed as in solved Ex., Art. 24.]

8. Prove that the line joining $(1, -1)$ and $(0, 1)$ bisects the line joining $(1, 2)$ and $(-3, 4)$.

9. Find the equation of the straight line passing through the points (3, 3) and (7, 6). What is the length of the portion of the line intercepted between the axes of co-ordinates ? [*P. U. 1950*]

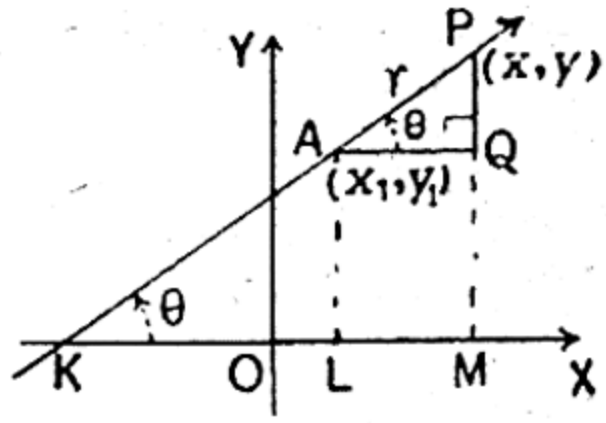
III. Distance form.

30. To obtain the equation of a straight line (drawn through a given point (x_1, y_1) and making a given angle θ with the x-axis) in the form

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r,$$

where r is the distance of the point (x, y) from (x_1, y_1).*

Let A be the pt. (x_1, y_1) and KA the line meeting the x-axis in K, so that $\angle XKA=\theta$.



(*i*) Let P(x, y) be *any* pt. on the line, so that AP$=r$. From A, P draw AL, PM $\perp$s on the x-axis and AQ $\perp$ on PM.

Then $\angle QAP=\angle XKA$ [Corresponding $\angle$s]
$= \theta$.

(*ii*) From the rt. $\angle$d $\triangle$ AQP,

$$\cos\theta=\frac{AQ}{AP},\ \sin\theta=\frac{QP}{AP} \quad \ldots (1) \qquad [\textit{Note this step}]$$

But $AQ=LM=OM-OL=x-x_1$,
$AP=r$,
$QP=MP-MQ=MP-LA=y-y_1$,

$\therefore$ from (1),

$$\cos\theta=\frac{x-x_1}{r},\ \sin\theta=\frac{y-y_1}{r}$$

or $$\frac{\mathbf{x-x_1}}{\mathbf{\cos\theta}}=\frac{\mathbf{y-y_1}}{\mathbf{\sin\theta}}=\mathbf{r} \quad \ldots (2),$$

which is the required equation. †

N. B. Sign of r. If P is *above* A (as in the Fig.), r is positive, and if it is below, r is negative.

Cor. 1. To express the co-ordinates of any point on the line $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$ in terms of one variable (called the *parameter*).

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \quad \ldots (1)$$

$\therefore$ from the first and third members of (1),
$x-x_1=r\cos\theta$, $\therefore$ $x=x_1+r\cos\theta$.
Again from the second and third members of (1),
$y-y_1=r\sin\theta$, $\therefore$ $y=y_1+r\sin\theta$
$\therefore$ $\mathbf{x=x_1+r\cos\theta}$, $\mathbf{y=y_1+r\sin\theta}$,

which are the required co-ordinates.‡

Cor. 2. Any point on the line. *Any point on the line* $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$ *is* $(\mathbf{x_1+r\cos\theta, y_1+r\sin\theta})$.

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*[*P. U. 1957*]

†Strictly speaking, the *equation of the line* consists of the *first two* members, the third member being *another variable*, namely, the distance of (x, y) from (x_1, y_1).

‡In terms of one variable r.

[Rule to find any point on a line (equation in the distance form): *In the equation of the line put each member=r, and find the values of x and y. These are the co-ordinates of the required point.*]

Note 1. Distance form. Since in the equation $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$, r is the *distance* of the point (x, y) from (x_1, y_1), this form of the equation of a line may be called the **distance form.**

Note 2. From the first two members of (2), Art. 30,

$$y-y_1=\tan\theta.\ (x-x_1)=m(x-x_1).$$ [*Cf.* Art. 28.]

****Note 3. Parameter. Def.** A variable in terms of which the co-ordinates of *any* point on a line are expressed, is called a **parameter,** and the co-ordinates are called the **parametric co-ordinates.**

Thus $(x_1+r\cos\theta,\ y_1+r\sin\theta)$ are the **parametric co-ordinates** of any point on the line, r being the **parameter.**

****Note 4. What is the use of the co-ordinates $(x_1+r\cos\theta,\ y_1+r\sin\theta)$?**

The use of the co-ordinates $(x_1+r\cos\theta,\ y_1+r\sin\theta)$ is that they enable us to find the *length* of a straight line drawn from a given point in a given direction to meet a given curve. (See the following solved Ex.)

Example. *Find the equation of the line through the point P(2. 3) and making an angle of 45° with the axis of x and determine the length intercepted on it between the point P and the line* $x+y+1=0$. [*D. U. 1931*]

(*i*) Here $\theta=45°$, $\therefore$ the equation of the line thro' (2, 3), in the distance form, is $\frac{x-2}{\cos 45°}=\frac{y-3}{\sin 45°}=r$ $\left[\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r\right]$

or $$\frac{x-2}{\frac{1}{\sqrt{2}}}=\frac{y-3}{\frac{1}{\sqrt{2}}}=r \quad \ldots(1)$$

or, from the first two members, $x-2=y-3$ or $x-y+1=0$.*

(*ii*) From (1), *any* pt. on the line is

$$\left(2+\frac{r}{\sqrt{2}},\ 3+\frac{r}{\sqrt{2}}\right)\dagger.$$ (Values of x, y)

If it lies on $x+y+1=0$, then

$$\left(2+\frac{r}{\sqrt{2}}\right)+\left(3+\frac{r}{\sqrt{2}}\right)+1=0 \text{ or } r\sqrt{2}+6=0,\ \therefore\ r=-3\sqrt{2}.$$

$\therefore$ changing the sign, distance or length *in magnitude* $=3\sqrt{2}$.

EXAMPLES

1. Obtain the equation of a straight line in the form $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$, clearly stating the significance of (x_1, y_1), θ and r.

[*P. U. 1957*]

**Or*, at once, $m=\tan 45°=1$, $\therefore$ the equation of the line is
$y-3=1\ (x-2)$. $[y-y_1=m\ (x-x_1)]$

†*Or*, at once, $(2+r\cos 45°,\ 3+r\sin 45°)$ $[x_1+r\cos\theta,\ y_1+r\sin\theta]$

2. *A straight line is drawn through the point $P(1, \sqrt{3})$ at an angle of 60° to the x-axis. Find the length of the line measured from P to the point where it meets the line $x+\sqrt{3}y=8$.*

[**Hint.** The equation of the line, in the distance form, is

$$\frac{x-1}{\cos 60^\circ}=\frac{y-\sqrt{3}}{\sin 60^\circ}=r$$

or
$$\frac{x-1}{\frac{1}{2}}=\frac{y-\sqrt{3}}{\frac{\sqrt{3}}{2}}=r.$$

$\therefore$ *any* pt. on the line is $\left(1+\frac{r}{2}, \sqrt{3}+\frac{\sqrt{3}\,r}{2}\right)$. If it lies on $x+\sqrt{3}y=8$, find the value of r.]

3. Find the distance of the line $3x-y=0$ from the point (4, 1) along a line making an angle of 135° with the x-axis. [*M. U. 1941*]

4. *A line is drawn through the point P(3, 4) inclined to the x-axis at an angle of $3\pi/4$. Find its equation, and find also the co-ordinates of the two points on it on opposite sides of P at a distance $\sqrt{2}$ from it.* [*P. U.*]

**5. Find the direction in which a straight line must be drawn through the point (1, 2), so that its point of intersection with the line $x+y=4$ may be at a distance $\frac{1}{3}\sqrt{6}$ from this point. [*P. U. 1943*]

[**Hint.** Let θ be the required inclination of the line to the x-axis. Then the equation of the line, in the distance form, is

$$\frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=r. \quad \text{Here } r=\tfrac{1}{3}\sqrt{6},$$

$\therefore$ $(1+\frac{1}{3}\sqrt{6}\cos\theta,\ 2+\frac{1}{3}\sqrt{6}\sin\theta)$ lies on $x+y=4$,

$\therefore$ $1+\frac{1}{3}\sqrt{6}\cos\theta+2+\frac{1}{3}\sqrt{6}\sin\theta=4$ or $\frac{\sqrt{6}}{3}(\sin\theta+\cos\theta)=1$

or $\sin\theta+\cos\theta=\frac{3}{\sqrt{6}}$. Dividing thro' out by $\sqrt{(1)^2+(1)^2}$,*

i.e., by $\sqrt{2}$, $\sin(\theta+45^\circ)=\frac{3}{\sqrt{12}}=\frac{3}{2\sqrt{3}}=\frac{\sqrt{3}}{2}=\sin 60^\circ$ or $\sin(180^\circ-60^\circ)$

$=\sin 60^\circ$ or $\sin 120^\circ$. $\therefore$ $\theta+45^\circ=60^\circ$ or 120°.]

Identical lines

31. Comparing coefficients. *If the equations*

$$Ax+By+C=0$$

and

$$A'x+B'y+C'=0$$

represent the same straight line, then

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}.$$

[In words : **If two equations represent the same line, ratio of coefficients of x = ratio of coefficients of y = ratio of constant terms.**]

****Proof.** The equations are

$$Ax+By+C=0 \quad \ldots(1)$$

*See the author's *New Intermediate Plane Trigonometry (6th Edition)* Art. 81,(*d*), *Rule*.

$$A'x+B'y+C'=0 \quad \ldots(2)$$

Writing the equations in the form $y=mx+c$,

from (1), $By=-Ax-C$, $\therefore\ y=-\dfrac{A}{B}\,x-\dfrac{C}{B}$ ...(3)

and from (2), $B'y=-A'x-C'$, $\therefore\ y=-\dfrac{A'}{B'}\,x-\dfrac{C'}{B'}$ ...(4)

Now the equations (3) and (4) represent the same st. line [*Given*]
$\therefore$ the slope is the same,

i.e., $$-\frac{A}{B}=-\frac{A'}{B'} \text{ or } \frac{A}{A'}=\frac{B}{B'} \quad \ldots(5)$$

Also the intercept on the y-axis is the same

$\therefore$ $$-\frac{C}{B}=-\frac{C'}{B'} \text{ or } \frac{B}{B'}=\frac{C}{C'} \quad \ldots(6)$$

From (5) and (6), $\dfrac{A}{A'}=\dfrac{B}{B'}=\dfrac{C}{C'}$.

This is called **comparing coefficients.**
It will be found extremely useful in later chapters.

EXAMPLE

If $Ax+By=C$ and $x\cos\alpha+y\sin\alpha=p$ represent the same line, find p in terms of A, B, C. [*P. U.*]

[**Hint.** Comparing coefficients, $\dfrac{A}{\cos\alpha}=\dfrac{B}{\sin\alpha}=\dfrac{C}{p}$,

$\therefore\ \cos\alpha=\dfrac{A}{C}\,p$, $\sin\alpha=\dfrac{B}{C}\,p$. Square and add. Use $\cos^2\alpha+\sin^2\alpha=1$.]

Tracing a line

32. *To trace the straight line* $Ax+By+C=0$.

The equation of the line is $Ax+By+C=0$ ... (1)

(*i*) Putting $y=0$ in (1), $Ax+C=0$ or $x=-\dfrac{C}{A}$,

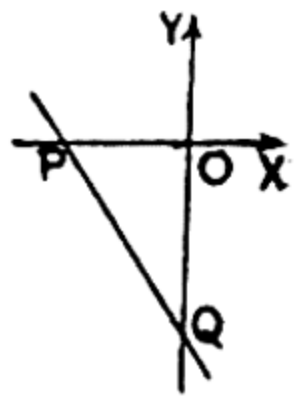
which is the intercept on the x-axis. [Art. 26, Cor.]

Again putting $x=0$ in (1), $By+C=0$ or $y=-\dfrac{C}{B}$,

which is the intercept on the y-axis.

(*ii*) Measure OP along the x-axis and $=-\dfrac{C^*}{A}$,

measure OQ along the y-axis and $=-\dfrac{C}{B}$, observing the Rule for signs of intercepts [Art. 19, (*b*)].



Join PQ, and produce it both ways. This is the required line.

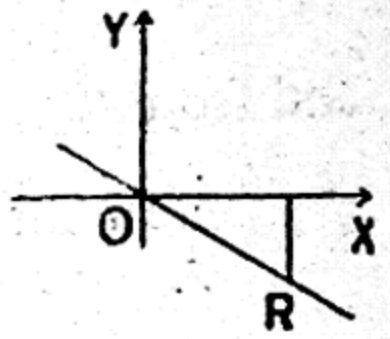
Exceptional cases.

Case I. No constant term. If the equation contains no constant term, it becomes $Ax+By=0$ ... (1)

*In drawing the Figs. A, B, C have all been taken to be *positive*.

(*i*) Putting $y=0$ in (1), $x=0$, $\therefore$ (0, 0), *i.e.*, the origin is one pt. on the line.

Again from (1), $y=-\frac{A}{B}x$. Putting $x = B$ (to avoid fractions), $y=-A$. $\therefore$ (B, $-$A) is another pt. on the line.

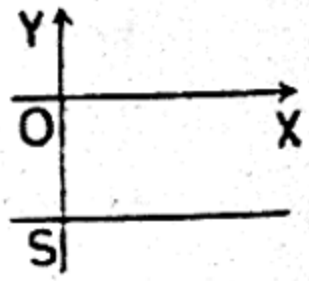


(*ii*) Plot the pt. R(B, $-$A).

Join OR, and produce it both ways. This is the required line.

Case II. No term in x. If the equation contains no term in x, it becomes $By+C=0$ ... (1)

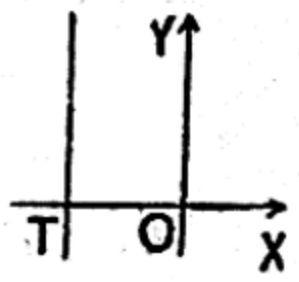
(*i*) From (1), $y=-\frac{C}{B}$. $\therefore$ the line is $\parallel$ to the x-axis at a distance $-\frac{C}{B}$ from it (Art. 15).



(*ii*) Measure OS along the y-axis and $=-\frac{C}{B}$. Thro' S draw a st. line $\parallel$ to the x-axis. This is the required line.

Case III. No term in y. If the equation contains no term in y, it becomes $Ax+C=0$ ... (1)

(*i*) From (1), $x=-\frac{C}{A}$. $\therefore$ the line is $\parallel$ to the y-axis and at a distance $-\frac{C}{A}$ from it (Art. 16).



(*ii*) Measure OT along the x-axis and $=-\frac{C}{A}$.

Thro' T draw a st. line $\parallel$ to the y-axis. This is the required line.

Example. Trace the following straight lines :

(1) $2x-3y+6=0$, (2) $x-2y=0$, (3) $x+5=0$, (4) $y+3=0$.

(1) The equation of the line is $2x-3y+6=0$ ... (1)

(*i*) Putting $y=0$ in (1), $2x+6=0$ or $x=-3$, which is the intercept on the x-axis.

Again putting $x=0$ in (1), $-3y+6=0$ or $y=2$, which is the intercept on the y-axis.

(*ii*) [**Scale.** 1 small division $=1$.]

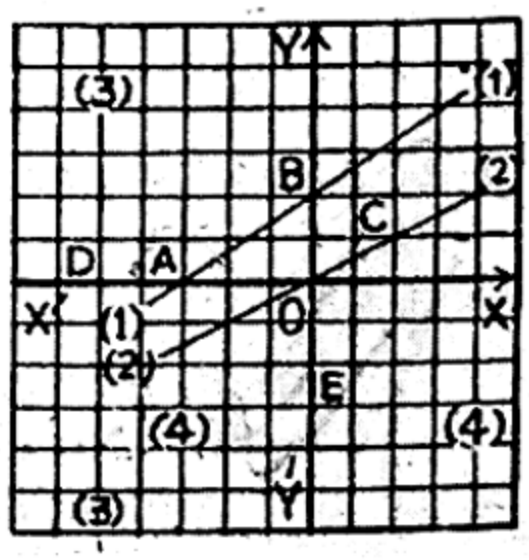
Measure OA along the x-axis to the *left* of O and $=3$ in magnitude, measure OB along the y-axis *above* O and $=2$.

Join AB, and produce it both ways. This is the required line marked (1) in the Fig.

(2) The equation of the line is $x-2y=0$ ...(1)

(*i*) Putting $y=0$ in (1), $x=0$, $\therefore$ (0, 0), *i.e.*, the origin is one pt. on the line.

Again from (1), $y=\frac{x}{2}$. Putting $x=2$, $y=1$.



$\therefore$ (2, 1) is another pt. on the line.

(*ii*) Plot the pt. C(2, 1). Join OC, and produce it both ways. This is the required line marked (2) in the Fig.

(3) The equation of the line is $x+5=0$ ... (1)

(*i*) From (1), $x=-5$. $\therefore$ the line is $\parallel$ to the y-axis at a distance -5 from it.

(*ii*) Measure OD along the x-axis to the *left* of O and $=5$ in magnitude. Thro' D draw a st. line $\parallel$ to the y-axis. This is the required line marked (3) in the Fig.

(4) The equation of the line is $y+3=0$ ... (1)

(*i*) From (1), $y=-3$. $\therefore$ the line is $\parallel$ to the x-axis at a distance -3 from it.

(*ii*) Measure OE along the y-axis *below* O and $=3$ in magnitude. Thro' E draw a st. line $\parallel$ to the x-axis. This is the required line marked (4) in the Fig.

EXAMPLE

Draw the following lines :

(*i*) $5y=3x$ and (*ii*) $y=3x+12$.

(*iii*) $y+2=0$; (*iv*) $x-2=0$. *[Ag. U. 1932]*

SECTION II

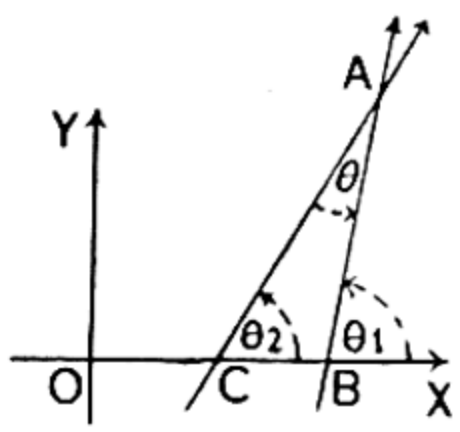
ANGLE BETWEEN TWO STRAIGHT LINES

33-I. Angle formula I. Slope forms. To find the angle between the two straight lines

$$\mathbf{y=m_1x+c_1 \text{ and } y=m_2x+c_2}^{*}.$$

Let BA, CA be the lines $y=m_1x+c_1$, $y=m_2x+c_2$ meeting the x-axis in B, C. Let $\angle XBA=\theta_1$, $\angle XCA=\theta_2$ so that $\tan\theta_1=m_1$, $\tan\theta_2=m_2$.

Let θ be the required angle CAB between the lines.



Then from the Fig.,

$$\theta=\theta_1-\theta_2 \qquad [\because\ \theta_1=\theta+\theta_2]$$

$$\therefore \quad \tan\theta=\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$

or $$\mathbf{\tan\theta=\frac{m_1-m_2}{1+m_1m_2}}, \quad \therefore\ \theta=\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}.$$

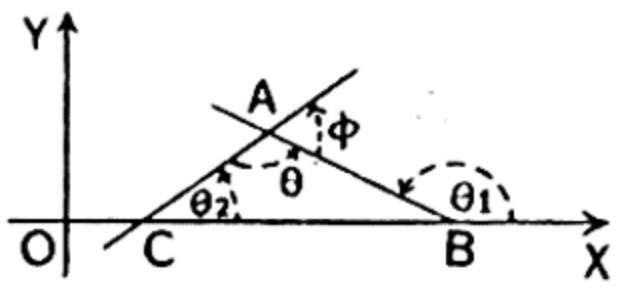
Note 1. See the Fig. θ is the angle between the lines, measured *positively from* $y=m_2x+c_2$ to $y=m_1x+c_1$, and lies between 0° and 180°.

Note 2. Acute and obtuse angles between two lines.

In any numerical example

(*i*) if the angle formula $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$ gives a *positive* result, it is the tangent of the *acute* angle between the lines ; but

(*ii*) if the angle formula gives a *negative* result, it is the tangent of the *obtuse* angle between the lines. In this case, if ϕ is the acute angle between them, then $\phi=\pi-\theta$



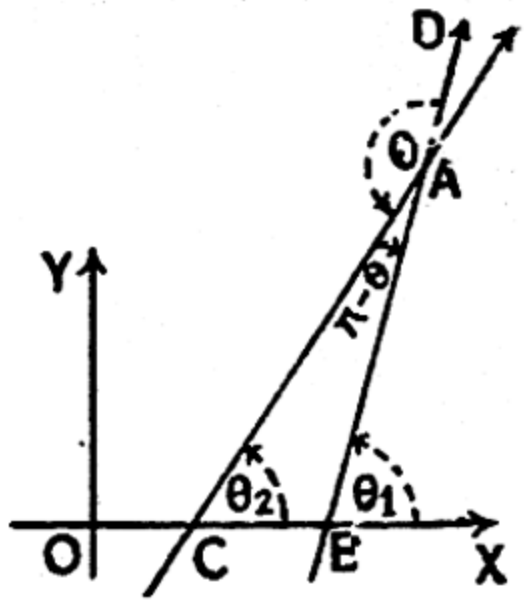
$$\therefore\ \tan\phi=\tan(\pi-\theta)=-\tan\theta=-\frac{m_1-m_2}{1+m_1m_2},$$

*[J. & K. U. 1956]

i.e., we must *change the sign in the value of* $\tan\theta\left(=\frac{m_1-m_2}{1+m_1m_2}\right)$ *to get* $\tan\phi$, *the tangent of the acute angle between the lines.* (*Note this step*)

(See Ex. 3 following.)

Note 3. Complete angle formula. If θ (instead of being the required $\angle$CAB between the lines as in the Fig. of Art. 33-I) is the required $\angle$DAC between the lines as in the adjoining Fig., then



$$\pi-\theta=\theta_1-\theta_2 \qquad [\because\ \theta_1=(\pi-\theta)+\theta_2]$$

$$\therefore\ \tan(\pi-\theta)=\tan(\theta_1-\theta_2)$$

or $$-\tan\theta=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}=\frac{m_1-m_2}{1+m_1m_2}$$

or $$\tan\theta=-\frac{m_1-m_2}{1+m_1m_2},$$

$\therefore$ combining this with the angle formula of Art. 33-I,

$$\mathbf{\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2}.}$$

[**Aid to memory.** For the complete angle formula (like the complete area formula) take the double sign ($\pm$) with the ordinary angle formula.]

Note 4. When to use the complete angle formula. The complete angle formula is used when the angle between the lines is *given*.

(See Art. 38.)

33-II. Angle formula II. General forms. To find the angle between the two straight lines

$$\mathbf{a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0.}*$$

The equations of the lines are $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$.

Here $$m_1=-\frac{a_1}{b_1},$$ $\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y}\text{ (Art. 25, Cor.)}\right]$

$$m_2=-\frac{a_2}{b_2},$$

$\therefore$ if θ is the angle between the lines, then

$$\mathbf{\tan\theta=\frac{\left(-\frac{a_1}{b_1}\right)-\left(-\frac{a_2}{b_2}\right)}{1+\left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}} \qquad \left[\frac{m_1-m_2}{1+m_1m_2}\right]$$

$$=\frac{\frac{a_2}{b_2}-\frac{a_1}{b_1}}{1+\frac{a_1a_2}{b_1b_2}}=\mathbf{\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}}$$

$$\therefore\qquad \theta=\tan^{-1}\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$

[Note. **When to use this angle formula II. In numerical examples, the slopes of the two lines should first be found and then the angle formula I :

*[*J. & K. U. 1951*]

$\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$...(I) should be used, except when m_1 *or* $m_2=\infty$,

i.e., when the equation of one of the lines is of the form $x=a$, *in which case the angle formula I fails and the angle formula II :*

$\tan\theta=\dfrac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}$ *should be used.* (See Misc. Ex. 10, Chap. III.)

Example. *Find the angle between the lines*

$y=x\sqrt{3}+1$ and $x-\sqrt{3}y+2=0$.

The equations of the lines are $\sqrt{3}x-y+1=0$, $x-\sqrt{3}y+2=0$.

[R. H. S.'s zero (*Note this step*)]

Here $m_1=-\dfrac{\sqrt{3}}{-1}$ $\left[-\dfrac{\text{coeff. of } x}{\text{coeff. of } y}\text{ (Art. 25, Cor.)}\right]$

$=\sqrt{3}$,

$m_2=-\dfrac{1}{-\sqrt{3}}=\dfrac{1}{\sqrt{3}}$,

$\therefore$ if θ is the angle between the lines, then

$$\tan\theta=\frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{1+\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)} \qquad \left[\frac{m_1-m_2}{1+m_1m_2}\right]$$

$$=\frac{\frac{2}{\sqrt{3}}}{2}=\frac{1}{\sqrt{3}}=\tan 30^\circ$$

$\therefore$ $\theta=30^\circ$.

EXAMPLES

1. Find the angle between the straight lines :

(*i*) $y\sqrt{3}-x+1=0$ and $x+y\sqrt{3}-1=0$;

(*ii*) $ax+by+c=0$ and $(a+b)x-(a-b)y=0$;

(*iii*) $x+2y=3$ and $2x=y-3$;

(*iv*) $x-2y=3$ and $2x+3y=4$.

2. Find the tangent of the angle between the two straight lines whose intercepts on the axes are a, b and a', b' respectively. [*C.U.*]

3. Find the acute angle between the lines $2x-y+3=0$ and $x+y-2=0$.

[**Hint.** It will be found that $\tan\theta=-3$ ... (1)

If ϕ is the acute angle between the lines, then changing the sign in (1) [Art. 33-I, Note 2, end],

$\tan\phi=3$, $\therefore$ $\phi=\tan^{-1}3$.]

4. Find the acute angle between the lines $3x+y-7=0$ and $x+2y+9=0$.

[*D. U. 1944*]

5. *Find the angle between the lines* $x-2y=3$ *and* $4x-y=6$.

[**Note. Important.** *"The angle between two lines"* means, for definiteness, "the *acute* angle between two lines".]

6. *Find the angle between the lines* $3y=5x+4$ *and* $4x+y=5$.

33-III. Perpendicular forms. To find the angle between the straight lines $x \cos \alpha_1+y \sin \alpha_1=p_1$ and $x \cos \alpha_2+y \sin \alpha_2=p_2$.

The equations of the lines are $x \cos \alpha_1+y \sin \alpha_1-p_1=0$, $x \cos \alpha_2+y \sin \alpha_2-p_2=0$.

[R. H. S.'s zero (*Note this step*)]

Here $m_1=-\dfrac{\cos \alpha_1}{\sin \alpha_1}$ $\left[-\dfrac{\text{coeff. of } x}{\text{coeff. of } y} \text{ (Art. 25, Cor.)}\right]$

$=-\cot \alpha_1$,

$m_2=-\dfrac{\cos \alpha_2}{\sin \alpha_2}=-\cot \alpha_2$,

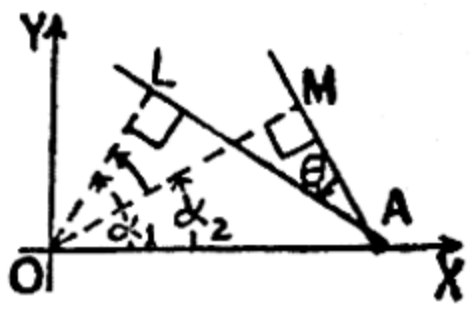
$\therefore$ if θ is the angle between the lines, then

$$\tan \theta=\frac{(-\cot \alpha_1)-(-\cot \alpha_2)}{1+(-\cot \alpha_1)(-\cot \alpha_2)} \qquad \left[\frac{m_1-m_2}{1+m_1m_2}\right]$$

$$=\frac{\cot \alpha_2-\cot \alpha_1}{1+\cot \alpha_1 \cot \alpha_2}=\frac{\dfrac{1}{\tan \alpha_2}-\dfrac{1}{\tan \alpha_1}}{1+\dfrac{1}{\tan \alpha_1}\cdot\dfrac{1}{\tan \alpha_2}}$$

$$=\frac{\tan \alpha_1-\tan \alpha_2}{1+\tan \alpha_1 \tan \alpha_2}=\tan (\alpha_1-\alpha_2), \qquad \therefore\ \theta=\alpha_1-\alpha_2.$$

Another method (Geometrically). Let AL, AM be the lines $x \cos \alpha_1+y \sin \alpha_1=p_1$, $x \cos \alpha_2+y \sin \alpha_2=p_2$, and OL, OM the $\perp$s from the origin O upon them, so that $\angle XOL=\alpha_1$, $\angle XOM=\alpha_2$. Let θ be the required angle MAL between the lines.



Then $\theta=\angle MAL=\angle MOL$ (in the same segment)

[$\because$ O, L, M, A lie on a circle, since $\angle OLA=\angle OMA$, each being $=1$ rt. $\angle$]

$=\angle XOL-\angle XOM=\alpha_1-\alpha_2$.

Parallel lines

34-I. Condition of parallelism. Slope forms. To find the condition that the two straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ may be parallel.

If the lines are $\parallel$, θ, the angle between them, $=0°$

$\therefore$ $\tan \theta=\tan 0°=0$,

i.e., $\dfrac{m_1-m_2}{1+m_1m_2}=0$ [Art. 33-I]

$\therefore$ $m_1-m_2=0$ or $m_1=m_2$, which is the required condition.

[In words : **If two lines are parallel, their slopes are equal.**]

Note. The Converse is also seen to be true by working the algebra backwards.

34-II. Condition of parallelism. General forms. To find the condition that the two straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ may be parallel.

The equations of the lines are $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$.

Here $m_1 = -\frac{a_1}{b_1}$, $\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y} \text{ (Art. 25, Cor.)}\right]$

$m_2 = -\frac{a_2}{b_2}$,

$\therefore$ if the lines are $\parallel$, $-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$ [$m_1 = m_2$ (Art. 34-I)]

[Cancel (-1)]

or $\frac{\mathbf{a_1}}{\mathbf{a_2}} = \frac{\mathbf{b_1}}{\mathbf{b_2}}$, which is the required condition.

[In words : **If two lines (equations in the general form) are parallel, ratio of the coefficients of x=ratio of the coefficients of y.**]

EXAMPLES

1. Show that the lines $y-2x=3$ and $2y=4x+5$ are parallel.
2. Prove that the straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half its length.

35. Any line parallel to a given line.

The equation of any line parallel to the line $Ax+By+C=0$ is

$$Ax+By+k=0.$$

Proof.* The equations of the lines are

$$Ax+By+C=0 \quad \ldots(1)$$
$$Ax+By+k=0 \quad \ldots(2)$$

From (1), $m_1 = -\frac{A}{B}$, $\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y} \text{ (Art. 25, Cor.)}\right]$

from (2), $m_2 = -\frac{A}{B}$,

$\therefore \quad m_1 = m_2, \quad \therefore$ the lines are $\parallel$. [Art. 34-I]

[Rule to write down the equation of any line parallel to a given line (equation in the general form) :

In the equation of the given line, change **only** *the constant term to a new constant k.*

Note. The value of k is found from the second condition satisfied by the line.]

Example. Find the equation of a line parallel to $3x+4y=12$ and passing through the point (4, 3). [*P. U. 1937*]

Method I. (*i*) The equation of *any* line $\parallel$ to $3x+4y-12=0$ is

$$3x+4y+k=0 \quad \ldots(1)$$ [Art. 35, Rule†]

***Or thus :** The equations of the lines are

$$Ax+By+C=0,\ Ax+By+k=0.$$

Ratio of the coeffs. of $x = \frac{A}{A} = 1$,

ratio of the coeffs. of $y = \frac{B}{B} = 1$.

$\therefore$ ratio of the coeffs. of x = ratio of the coeffs. of y

$\therefore$ the lines are $\parallel$. [Art. 34-II]

†**Explanation.** In the equation of the given line $3x+4y-12=0$, change only the constant term (-12) to a new constant $(+k)$, thus getting

$$3x+4y+k=0.$$

(*ii*) If it passes thro' (4, 3), then
$3(4)+4(3)+k=0$ or $24+k=0$, $\therefore\ k=-24$.

(*iii*) Substituting this value of k in (1),
$3x+4y-24=0$, which is the required equation.

Method II. (*i*) The equation of *any* line thro' (4, 3) is
$y-3=m(x-4)$ ...(1) $[y-y_1=m(x-x_1)]$

(*ii*) If it is $\parallel$ to $3x+4y-12=0$ (slope$=-\frac{3}{4}$), then
$m=-\frac{3}{4}$. $[m_1=m_2]$

(*iii*) Substituting this value of m in (1),
$y-3=-\frac{3}{4}(x-4)$ or $4y-12=-3x+12$
or $3x+4y-24=0$, which is the required equation.

[Two important methods to find the equation of the line parallel to a given line and passing through a given point.

Method I. (*i*) *Write down the equation of* **any** *line* $\parallel$ *to the given line.* [Art. 35, Rule]

(*ii*) *Make it pass thro' the given pt., and find the value of k.*

(*iii*) *Substitute this value of k in the equation of step* (*i*). *The result is the required equation.*

Method II. (*i*) *Write down the equation of* **any** *line passing thro' the given pt.* $[y-y_1=m(x-x_1)]$

(*ii*) *Make it* $\parallel$ *to the given line, and find the value of m.*

(*iii*) *Substitute this value of m in the equation of step* (*i*). *The result is the required equation.*]

EXAMPLES

1. The equation of the line parallel to $Ax+By+C=0$ and passing through (x_1, y_1) is $Ax+By=Ax_1+By_1$. [*B. U.*]

[****Rule to write down the equation of the line passing through a given point and parallel to a given line (equation in the general form):**

(*i*) *In the L. H. S. of the equation of the given line* (*R. H. S. being zero*), *delete the constant term. This is the L.H.S. of the required equation.*

(*ii*) *In the L.H.S. obtained above, substitute the co-ordinates of the given pt. This is the R.H.S. of the required equation.*]

2. Find the equation of the line parallel to $3x+4y-12=0$ and passing through (4, 3).

[****Shortest method.**

The equation of the line $\parallel$ to $3x+4y-12=0$ and passing thro' (4,3) is $3x+4y=3(4)+4(3)$ [Ex. 1, Rule*]
or $3x+4y=24$ or $3x+4y-24=0$.]

3. Find the equation of the line through (2, 3) parallel to $3x-4y+5=0$. [*P. U.*]

4. Find the equation of a straight line through (4, 5) parallel to $2x-3y-5=0$. [*D. U. 1947*]

***Explanation.** (*i*) In the L.H.S. of the equation of the given line $3x+4y-12=0$, delete the constant term (-12), thus getting $3x+4y$. This is the L. H. S. of the required equation.

(*ii*) In the L.H.S. obtained above, *i.e.*, $3x+4y$, substitute the co-ordinates of the given pt. (4, 3) thus getting $3(4)+4(3)$. This is the R. H. S. of the required equation.

5. *Find the equation of the line through (1, −1) parallel to the line joining (1, 3) and (2, −1).*

[**Short-cut. Hint.

The slope of the line passing thro' (1, 3) and (2, −1)

$$= \frac{-1-3}{2-1} \qquad \left(\frac{y_2-y_1}{x_2-x_1}\right)$$

$$= \frac{-4}{1} = -4.$$

Use Method II of solved Ex., Art. 35.]

6. Find the equation of the line through (−1, 1) parallel to the line joining (1, −2) and (−2, −3).

Perpendicular lines

36-I. Condition of perpendicularity. Slope forms. To find the condition that the two straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ may be perpendicular.

If the lines are $\perp$,

θ, the angle between them, $=90°$

$\therefore$ $\tan\theta=\tan 90°=\infty$,

i. e., $\dfrac{m_1-m_2}{1+m_1m_2}=\infty$

$\therefore$ the *denominator** $1+m_1m_2=0$

or $\mathbf{m_1m_2=-1}$

or $\mathbf{m_2=-\dfrac{1}{m_1}}$, which is the required condition.

[In words : **If two lines are perpendicular, the product of their slopes $=-1$, or the slope of one line is the negative reciprocal of the slope of the other.**]

Note. The Converse is also seen to be true by working the algebra backwards.

***Important. If a fraction $=\infty$, its denominator $=0$.**

36-II. Condition of perpendicularity. General forms. To find the condition that the two straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ may be perpendicular.

Here $m_1=-\dfrac{a_1}{b_1}$, $\left[-\dfrac{\text{coeff. of } x}{\text{coeff. of } y} \quad \text{(Art. 25, Cor.)}\right]$

$m_2=-\dfrac{a_2}{b_2}$,

$\therefore$ if the lines are $\perp$, $\left(-\dfrac{a_1}{b_1}\right)\left(-\dfrac{a_2}{b_2}\right)=-1$

[$m_1m_2=-1$ (Art. 36-I)]

or $a_1a_2=-b_1b_2$ or $\mathbf{a_1a_2+b_1b_2=0}$,

which is the required condition.

[In words : **If two lines (equations in the general form) are perpendicular,**

product of the coefficients of x + product of the coefficients of y = 0.]

Note. The Converse is also seen to be true by working the algebra backwards.

EXAMPLES

1. (*a*) Find the angle between the lines $ax+by+c=0$ and $a'x+b'y+c'=0$. [*B.H.U. 1947*]

(*b*) Deduce the conditions that the lines may be (1) parallel, (2) perpendicular to one another. [*B.H.U. 1947*]

[**Sol.** (*b*) (1) If the lines are $\parallel$,

θ, the angle between them, $=0°$

$\therefore$ $\tan\theta=\tan 0°=0$,

$$i.e.,\ \frac{a'b-ab'}{aa'+bb'}=0$$ [From part (*a*)]

$\therefore$ $a'b-ab'=0$ or $a'b=ab'$ or $\frac{a}{a'}=\frac{b}{b'}$,

which is the required condition.

(2) If the lines are $\perp$,

θ, the angle between them, $=90°$

$\therefore$ $\tan\theta=\tan 90°=\infty$,

$$i.e.,\ \frac{a'b-ab'}{aa'+bb'}=\infty$$ [From part (*a*)]

$\therefore$ the *denominator* $aa'+bb'=0$, which is the required condition.]

2. Show that the lines $3x+4y+7=0$ and $28x-21y+50=0$ are perpendicular to each other. [*P. U.*]

3. Prove that the line joining (1, 2) and (2, −2) is perpendicular to the line joining (8, 2) and (4, 1).

37. Any line perpendicular to a given line.

The equation of any line perpendicular to the line $Ax+By+C=0$ is $Bx-Ay+k=0$.

Proof.* The equations of the lines are -

$$Ax+By+C=0 \quad \ldots (1)$$
$$Bx-Ay+k=0 \quad \ldots (2)$$

From (1), $m_1=-\frac{A}{B}$, $\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y}\ \text{(Art. 25, Cor.)}\right]$

from (2), $m_2=-\frac{B}{-A}=\frac{B}{A}$,

$\therefore$ $m_1m_2=-\frac{A}{B}\cdot\frac{B}{A}=-1$, $\therefore$ the lines are $\perp$. [Art. 36-I]

[Rule to write down the equation of any line perpendicular to a given line (equation in the general form) :

(i) In the equation of the given line, interchange the coefficients of x and y, (ii) change the sign of one of them, and (iii) change the constant term to a new constant k.

***Or thus :** The equations of the lines are

$$Ax+By+C=0,\quad Bx-Ay+k=0.$$

Product of the coeffs. of x + product of the coeffs. of $y=A(B)+B(-A)=0$,

$\therefore$ the lines are $\perp$. [Art. 36-II]

Note. The value of k is found from the second condition satisfied by the line.]

Example. Find the equation of a straight line through the point (4, 5) perpendicular to the line $3x-2y+5=0$. [*P.U. 1942*]

Method I. (*i*) The equation of *any* line $\perp$ to $3x-2y+5=0$ is

$$2x+3y+k=0 \quad \ldots(1)$$ [Art. 37, Rule*]

(*ii*) If it passes thro' (4, 5), then

$$2(4)+3(5)+k=0 \quad \text{or} \quad 23+k=0, \quad \therefore \ k=-23.$$

(*iii*) Substituting this value of k in (1),

$2x+3y-23=0$, which is the required equation.

Method II. (*i*) The equation of *any* line thro' (4, 5) is

$$y-5=m(x-4) \quad \ldots(1)$$ [$y-y_1=m(x-x_1)$]

(*ii*) If it is $\perp$ to $3x-2y+5=0$ $\left(\text{slope}=-\frac{3}{-2}=\frac{3}{2}\right)$, then

$$m\left(\frac{3}{2}\right)=-1$$ [$m_1m_2=-1$]

$$\therefore \quad m=-\tfrac{2}{3}.$$

(*iii*) Substituting this value of m in (1),

$$y-5=-\tfrac{2}{3}(x-4) \quad \text{or} \quad 3y-15=-2x+8$$

or $2x+3y-23=0$, which is the required equation.

EXAMPLES

1. The equation of the line perpendicular to $Ax+By+C=0$ and passing through (x_1, y_1) is $Bx-Ay=Bx_1-Ay_1$. [*B.U.*]

[****Rule to write down the equation of the line passing through a given point and perpendicular to a given line (equation in the general form) :**

(i) In the L.H.S. of the equation of the given line (R.H.S. being zero), delete the constant term, interchange the coefficients of x and y, and change the sign of one of them. This is the L.H.S. of the required equation.

(ii) In the L.H.S. obtained above, substitute the co-ordinates of the given pt. This is the R.H.S. of the required equation.]

2. Find the equation of a line going through the origin and perpendicular to $5x+12y+13=0$. [*P.U.*]

[****Shortest method.**

The equation of the line $\perp$ to $5x+12y+13=0$ and passing thro' (0, 0) is

$$12x-5y=12(0)-5(0)$$ [Ex. 1, Rule†]

or $12x-5y=0$.]

***Explanation.** (*i*) In the equation of the given line $3x-2y+5=0$, interchange the coefficients 3 and -2 of x and y, thus getting $-2x+3y+5=0$, (*ii*) change the sign of one of them (-2), thus getting $2x+3y+5=0$, and (*iii*) change the constant term $(+5)$ to a new constant $(+k)$, thus getting $2x+3y+k=0$.

†**Explanation.** (*i*) In the L. H. S. of the equation of the given line $5x+12y+13=0$, delete the constant term $(+13)$, thus getting $5x+12y$, interchange the coefficients 5 and 12 of x and y, thus getting $12x+5y$, and change the sign of one of them $(+5)$, thus getting $12x-5y$. This is the L.H.S. of the required equation.

(*ii*) In the L.H.S. obtained above, *i.e.*, $12x-5y$, substitute the co-ordinates of the given pt. (0, 0), thus getting $12(0)-5(0)$. This is the R.H.S. of the required equation.

3. Find the equation of the straight line (*i*) through the point (3, 4) and perpendicular to the line $3y-2x=1$. [*B.H.U.*]

(*ii*) through the point (x', y') and perpendicular to $x'y+xy'=a^2$.

4. Obtain the equation of a line (*i*) which passes through (h, k) and which is perpendicular to $lx+my-1=0$. [*P.U. 1945*]

(*ii*) which passes through the point (a, b) and is at right angles to the line $px-qy-r=0$. [*P.U. 1945S*]

[**Hint.** (*i*) The equation of *any* line $\perp$ to $lx+my-1=0$ is

$$mx-ly+p=0 \quad ...(1)$$

Note. Why to use p (and not k). In order to avoid confusion with the given k in (h, k) we have taken the equation of *any* line $\perp$ to $lx+my-1=0$ as $mx-ly+p=0$ (and *not* $mx-ly+k=0$).]

5. *Given the triangle $A(10, 4)$, $B(-4,9)$, $C(-2, -1)$, find the equation of the altitude through B.* [*D.U. 1942*]

6. Write down the equation of the perpendicular bisector of the line joining the points (1, 1), (2, 3). [*B.U.*]

38. Lines through a given point inclined at a given angle to a given line.

Example 1. Find the equations of the two straight lines through the point (2, −1) making an angle of 45° with the line $6x+5y-1=0$. [*P.U. 1933S*]

(*i*) The equation of *any* line thro' (2, −1) is

$$y+1=m(x-2) \quad ...(1) \qquad [\ y-y_1=m(x-x_1)\ ]$$

(*ii*) If it makes an angle of 45° with $6x+5y-1=0$ $\left(\text{slope} = -\frac{6}{5}\right)$,

$$\text{then} \quad \tan 45° = \pm \frac{m-\left(-\frac{6}{5}\right)}{1+m\left(-\frac{6}{5}\right)} \qquad \left[\ \pm\frac{m_1-m_2}{1+m_1m_2} \text{(Art. 33-I, Note 4)}\ \right]$$

$$\therefore \qquad 1 = \pm \frac{5m+6}{5-6m} \qquad ...(2)$$

(I) Taking the +ve sign in (2),

$$5-6m=5m+6 \quad \text{or} \quad -11m=1 \quad \text{or} \quad m=-\frac{1}{11}.$$

(II) Taking the −ve sign in (2),

$$5-6m=-5m-6 \quad \text{or} \quad -m=-11 \quad \text{or} \quad m=11.$$

(*iii*) Substituting these values of m in (1),

(I) $y+1=-\frac{1}{11}(x-2)$ or $11y+11=-x+2$ or $x+11y+9=0$;

(II) $y+1=11(x-2) \quad =11x-22$ or $11x-y-23=0$.

$\therefore$ $x+11y+9=0$, $11x-y-23=0$, which are the required equations.

Note. *To find the equations of the two lines passing thro' a given point and making a given angle with a given line use the* **complete angle formula**.

****Example 2. Find the equations of the straight lines through the point (h, k) making a given angle α with the straight line $y=mx+c$.** [*J. & K. U. 1951*]

(*i*) The equation of *any* line thro' (h, k) is $y-k=M^*(x-h)$ ...(1)

(*ii*) If it makes an angle α with $y=mx+c$, then

$$\tan\alpha = \pm\frac{M-m}{1+Mm} \quad ...(2) \qquad \left[\ \pm\frac{m_1-m_2}{1+m_1m_2}\ \right]$$

(I) Taking the +ve sign in (2),

$\tan\alpha+Mm\tan\alpha=M-m$ or $M(m\tan\alpha-1)=-m-\tan\alpha$

$$\therefore \quad M=\frac{-m-\tan\alpha}{m\tan\alpha-1}=\frac{m+\tan\alpha}{1-m\tan\alpha}.$$

(II) Taking the −ve sign in (2),

$\tan\alpha+M\,m\tan\alpha=-M+m$

or $M(1+m\tan\alpha)=m-\tan\alpha$, $\therefore$ $M=\dfrac{m-\tan\alpha}{1+m\tan\alpha}$.

(*iii*) Substituting these values of M in (1),

(I) $y-k=\dfrac{m+\tan\alpha}{1-m\tan\alpha}(x-h)$

(II) $y-k=\dfrac{m-\tan\alpha}{1+m\tan\alpha}(x-h)$

} which are the required equations.

EXAMPLES

1. Find the equations of the two lines through the point (4, 5) which make an acute angle of 45° with the line $2x-y+7=0$.

[*Pesh. U. 1953*]

2. Find the equations to the straight lines passing through the point (3, −2) and inclined at 60° to the straignt line $x\sqrt{3}+y=1$.

[*P. U. 1944*]

3. Show that the equation of the line through the origin making an angle θ with the line $y=mx+c$ is $\dfrac{y}{x}=\dfrac{m+\tan\theta}{1-m\tan\theta}$. [*P. U. 1951*]

SECTION III

INTERSECTION OF STRAIGHT LINES

39. An important fact. *The co-ordinates of the points of intersection of two curves satisfy the equations of both curves.*

[∵ a pt. of intersection lies on both curves]

Hence the

Rule to find the points of intersection of two curves whose equations are given :

Solve the equations of the two curves simultaneously as in Algebra. The corresponding values of x and y are the required co-ordinates of the pts. of intersection.

40. Point of intersection of two lines. To find the co-ordinates

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***Why to use M (and not m).** In order to avoid confusion with the given m in $y=mx+c$, we have taken the equation of *any* line thro' (h, k) as $y-k=M(x-h)$ [and *not* $y-k=m(x-h)$].

of the point of intersection of the straight lines

$$\mathbf{a_1x+b_1y+c_1=0}$$

and

$$\mathbf{a_2x+b_2y+c_2=0.}$$

The equations of the lines are

$$a_1x+b_1y+c_1=0 \quad ...(1)$$
$$a_2x+b_2y+c_2=0 \quad ...(2)$$

Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\begin{matrix} x & y & 1 \\ \overbrace{} & \overbrace{} & \overbrace{} \\ b_1 \times c_1 & \times a_1 & \times b_1 \\ b_2 \quad c_2 & \quad a_2 & \quad b_2 \end{matrix} \qquad \text{[Art. 23, Rule]}$$

$$\therefore \frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

$$\therefore \quad x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\ y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1},$$

which are the required co-ordinates of the pt. of intersection.

[**Rule to find the point of intersection of two lines whose equations are given :**

Solve the equations of the two lines simultaneously as in Algebra. The values of x and y are the required co-ordinates of the pt. of intersection.]

EXAMPLES

1. *Find the point of intersection of the lines*

$$x-y=1 \text{ and } 2x-3y+1=0.$$

[**Sol.** The equations of the lines are

$$x-y-1=0 \quad ...(1)$$
$$2x-3y+1=0 \quad ...(2)$$

Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\begin{matrix} x & y & 1 \\ \overbrace{} & \overbrace{} & \overbrace{} \\ -1 \times -1 & \times 1 & \times -1 \\ -3 \quad 1 & \quad 2 & \quad -3 \end{matrix} \qquad \text{[Art. 23, Rule]}$$

$$\therefore \quad \frac{x}{-1-3}=\frac{y}{-2-1}=\frac{1}{-3-(-2)}$$

or $$\frac{x}{-4}=\frac{y}{-3}=\frac{1}{-1}, \quad \therefore\ x=\frac{-4}{-1}=4,\ y=\frac{-3}{-1}=3.$$

$\therefore$ the pt. of intersection is (4, 3).

[**Check.** The co-ordinates of the pt. of intersection (4, 3) satisfy both the equations $x-y-1=0$ and $2x-3y+1=0$, thus $4-3-1=0$ or $0=0$, and $2(4)-3(3)+1=0$ or $0=0$.]

2. Find the co-ordinates of the points of intersection of the straight lines

(*i*) $\mathbf{2x-3y-10=0}$ **and** $\mathbf{3x+7y+8=0.}$

(*ii*) $\mathbf{\frac{x}{a}+\frac{y}{b}=1}$ **and** $\mathbf{\frac{x}{b}+\frac{y}{a}=1.}$ [*C.U.*]

3. Foot of the perpendicular. Find the co-ordinates of the foot

of the perpendicular from the point (2, 3) on the line $y=3x+4$.

[*A. U. 1946*]

4. Find the equation of the line joining the feet of the perpendiculars from the origin to the lines $x+5y=13$ and $5x+y=13$.

[*D. U. 1940*]

5. Orthocentre. Find the co-ordinates of the orthocentre of the triangle whose angular points are (1, 0), (2, −4) and (−5, −2).

[*P.U. 1943S*]

[Rule to find the area of a triangle the equations of whose sides are given :

(*i*) *Take the equations two at a time, solve them simultaneously as in Algebra, and find the co-ordinates of the vertices, and then* (*ii*) *use the area formula* (*Art. 9*).]

6. Find the area of the triangle formed by the lines :

(*i*) $y-x=0$, and $y+x=0$, $x-c=0$; [*P. U. 1942S*]

(*ii*) $y+x=0$, $y=x+6$, $y=7x+5$. [*P. U. 1941*]

7. Show that the area of the triangle formed by the lines whose equations are $y=m_1x+c_1$, $y=m_2x+c_2$, and $x=0$ is $\frac{1}{2}\frac{(c_1-c_2)^2}{m_2-m_1}$.

[*C. U. 1955*]

8. Incentre. The line $x\cos\alpha+y\sin\alpha-p=0$ forms a triangle with the axes ; find the co-ordinates of the centre of the inscribed circle of this triangle. [*P.U.*]

9. Find the centre of the inscribed circle of the triangle the equations of whose sides are $y-15=0$, $12y+5x=0$ and $4y-3x=0$.

[*P. U.*]

Concurrency of lines

41. Condition of concurrency of three lines. To find the condition that three straight lines may meet in a point.*

Let the equations of the lines be

$$a_1x+b_1y+c_1=0 \quad \ldots(1)$$
$$a_2x+b_2y+c_2=0 \quad \ldots(2)$$
$$a_3x+b_3y+c_3=0 \quad \ldots(3)$$

[If the three lines meet in a pt. the pt. of intersection of two of the lines lies on the third.]

[Method of intersection.]

(*i*) Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\overbrace{\begin{matrix} b_1 \\ b_2 \end{matrix}}^{x} \times \overbrace{\begin{matrix} c_1 \\ c_2 \end{matrix}}^{y} \times \overbrace{\begin{matrix} a_1 \\ a_2 \end{matrix}}^{1} \times \begin{matrix} b_1 \\ b_2 \end{matrix} \qquad \text{[Art. 23, Rule]}$$

$$\therefore \frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1}$$

$\therefore x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$, $y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$, which are the co-ordinates of the pt. of intersection of (1) and (2).

(*ii*) Substituting these co-ordinates in (3),

*[*J. & K. U. 1957*]

$$a_3\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)+b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)+c_3=0$$

or $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$,
which is the required condition.

[**Rule to prove that three given lines meet in a point :**

(*i*) *Solve the equations of two of the lines simultaneously as in Algebra, and find the co-ordinates of their pt. of intersection.*

(*ii*) *Substitute these co-ordinates in the equation of the third line, and show that it is satisfied by them.*]

Example. Prove that the lines $x-y=6$, $4y-3x+22=0$ and $6x+5y+8=0$ are concurrent. [*P. U. 1935S*]

The equations of the lines are $x-y-6=0$* ...(1)

$3x-4y-22=0$ ...(2)

$6x+5y+8=0$ ...(3)

[**Method of intersection.**]

(*i*) Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\begin{array}{ccccc} x & & y & & 1 \\ -1 & \times & -6 & \times & 1 \\ -4 & & -22 & & 3 \end{array} \quad \begin{array}{c} \\ \times \\ \\ \end{array} \quad \begin{array}{c} \\ -1 \\ \\ -4 \end{array}$$

[Art. 23, Rule]

$$\therefore \frac{x}{22-24}=\frac{y}{-18+22}=\frac{1}{-4+3} \text{ or } \frac{x}{-2}=\frac{y}{4}=\frac{1}{-1}$$

$\therefore x=\frac{-2}{-1}=2$; $y=\frac{4}{-1}=-4$, which are the co-ordinates of the pt. of intersection of (1) and (2).

(*ii*) Substituting these co-ordinates in (3), we get $6(2)+5(-4)+8=0$, or $0=0$, which is true. $\therefore$ the lines are concurrent.

Note. The above method enables us not only to prove that the three lines meet in a point, but also to find *the co-ordinates of the point where they meet.*

EXAMPLES

1. Show that the three lines (*i*) $5x+3y=7$, $3x-4y=10$, and $x+2y=0$ meet in a point. [*C. U.*]

(*ii*) $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$, and $y=x$ meet in a point.

[*J. & K. U. 1952*]

2. For what value of k are the three lines $x-2y+1=0$, $2x-5y+3=0$, and $5x-9y+k=0$ concurrent ? [*P. U. 1947*]

3. What must be the value of a in order that the three lines $3x-y-2=0$, $5x+ay-3=0$, and $2x+y-3=0$, may meet in a point ?

4. Find the condition that the straight lines $y=m_1x+c_1$, $y=m_2x+c_2$, and $y=m_3x+c_3$ may meet in a point.

42. Test for concurrency of three lines in special cases.

If the equations of three straight lines are $a_1x+b_1y+c_1=0$,

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***Important.** *Before solving two equations by cross-multiplication, first put them in the form* $Ax+By+C=0$.

$a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$, and if three constants l, m, n can be found such that $l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0$ identically (i.e., $=0$ for **all** *values of x and y), then the three straight lines meet in a point.*

The equations of the lines are

$$a_1x+b_1y+c_1=0 \quad \ldots(1)$$
$$a_2x+b_2y+c_2=0 \quad \ldots(2)$$
$$a_3x+b_3y+c_3=0 \quad \ldots(3)$$

$\because$ $l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0$ for *all* values of x and y [Given]

$\therefore$ in particular,

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3x+c_3)=0 \quad \ldots(4)$$

for values of x and y which satisfy both (1) and (2).

$\therefore$ substituting from (1) and (2) in (4),

$$l(0)+m(0)+n(a_3x+b_3y+c_3)=0$$

or
$$a_3x+b_3y+c_3=0$$

$\therefore$ the co-ordinates of the pt. which satisfy both (1) and (2), also satisfy (3), $\therefore$ the pt. of intersection of the lines (1) and (2) lies on (3),

$\therefore$ the three lines meet in a pt.

Cor. Important. *If the equations of three straight lines are $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$ and $(a_1x+b_1y+c_1)+(a_2x+b_2y+c_2)+(a_3x+b_3y+c_3)=0$ identically, the three straight lines meet in a point.*

For, here $l=1$, $m=1$, $n=1$.

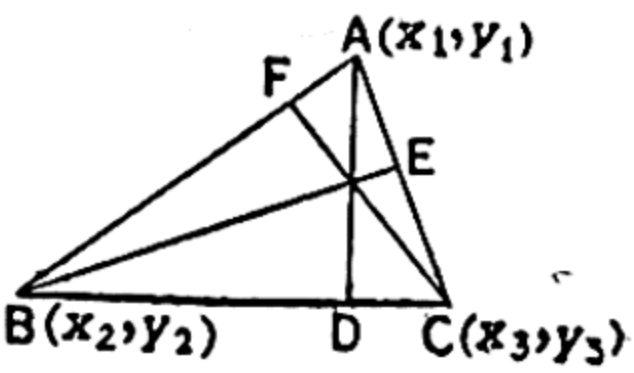
[In words : **Method of addition. If adding the L.H.S.'s of the equations of three lines (R.H.S.'s being zero) we get zero, the three lines meet in a point.**]

Example. Prove analytically that the three perpendiculars drawn from the vertices of a triangle upon the opposite sides meet in a point.

[*Bar. U. 1954*]

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle.

[To find the equation of the $\perp$ from A on BC.]



The equation of *any* line thro' $A(x_1, y_1)$ is $y-y_1=m(x-x_1)$ ... (1)

If it is $\perp$ to BC $\left(\text{slope} = \frac{y_3-y_2}{x_3-x_2}\right)$, then $m\left(\frac{y_3-y_2}{x_3-x_2}\right)=-1$,

$$\therefore \quad m=-\frac{x_3-x_2}{y_3-y_2}=-\frac{x_2-x_3}{y_2-y_3}.$$

Substituting this value of m in (1), the equation of the $\perp$ from A on BC is $y-y_1=-\frac{x_2-x_3}{y_2-y_3}(x-x_1)$

or $y(y_2-y_3)-y_1(y_2-y_3)=-x(x_2-x_3)+x_1(x_2-x_3)$

or $x(x_2-x_3)+y(y_2-y_3)-x_1(x_2-x_3)-y_1(y_2-y_3)=0$ ... (2)

Similarly* the equations of the ⊥s from B on CA and from C on AB are $x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$ ... (3)

and $x(x_1-x_2)+y(y_1-y_2)-x_3(x_1-x_2)-y_3(y_1-y_2)=0$ ... (4)

Adding the L. H. S.'s of (2), (3), (4) vertically, we get

$x(0)+y(0)-0-0=0$, ∴ the ⊥s meet in a pt. [Art. 42, Cor.]

[**Orthocentre. Def.** The point of concurrence of the *altitudes* of a triangle is called the **orthocentre** of the triangle.]

EXAMPLES

1. Prove that the lines

$$(b-c)x+(c-a)y+(a-b)=0,$$
$$(c-a)x+(a-b)y+(b-c)=0,$$
$$(a-b)x+(b-c)y+(c-a)=0,$$

are concurrent. [*P(P). U. 1952*]

2. Prove that the right bisectors of the sides of a triangle meet in a point.

3. The three straight lines joining the angular points of a triangle to the middle points of the opposite sides meet in a point. [*A. U. 1936*]

43. Any line through the intersection of two given lines.

To find the equation of any straight line through the point of intersection of the two straight lines $a_1x+b_1y+c_1=0$ *and* $a_2x+b_2y+c_2=0$.

Method I. [**K-method.**†]

The equations of the lines are

$$a_1x+b_1y+c_1=0 \quad \text{... (1)}$$
$$a_2x+b_2y+c_2=0 \quad \text{... (2)}$$

Consider the equation $a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0$ ... (3)

(*i*) It is an equation of the first degree in x and y

∴ it represents a st. line. [Art. 24]

(*ii*) The co-ordinates of the pt. which satisfy both (1) and (2), also satisfy (3) [∵ substituting from (1) and (2) in (3), we get $0+k(0)=0$, which is true]

∴ the pt. of intersection of the lines (1) and (2) lies on (3).

∴ (3) is the equation of a st. line passing thro' the pt. of intersection of the lines (1) and (2).

[**Aid to memory.** *The equation of any line thro' the pt. of intersection of two given lines is* **one line+k (other line)=0,** *where "one line" stands for "the L. H. S. of the equation of one line (R. H. S. being zero)" and so for the "other line".*]

Method II. [**Method of intersection.**]

The equations of the lines are

$$a_1x+b_1y+c_1=0 \quad \text{... (1)}$$
$$a_2x+b_2y+c_2=0 \quad \text{... (2)}$$

***How to write this step.** In equation (2) change x_1, x_2, x_3 and y_1, y_2, y_3 *cyclically, i.e.*, change x_1 to x_2, x_2 to x_3, x_3 to x_1 and y_1 to y_2, y_2 to y_3, y_3 to y_1, and we get (3). Similarly from (3) we get (4).

†[*P. U. 1956*]

(*i*) Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\begin{matrix} \overset{x}{\overbrace{}} & \overset{y}{\overbrace{}} & \overset{1}{\overbrace{}} \\ b_1 \times & c_1 \times & a_1 \times \\ b_2 & c_2 & a_2 \end{matrix} \quad \begin{matrix} b_1 \\ \times \\ b_2 \end{matrix}$$ [Art. 23, Rule]

$$\therefore \quad \frac{x}{b_1c_2-b_2c_1} = \frac{y}{c_1a_2-c_2a_1} = \frac{1}{a_1b_2-a_2b_1}$$

$$\therefore \quad x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \quad y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1},$$

which are the co-ordinates of the pt. of intersection.

(*ii*) The equation of *any* line thro' this pt. is

$$y-\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}=m\left(x-\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right), \quad [\ y-y_1=m(x-x_1) \quad \text{(Art. 28, Cor.)}\ ]$$

which is the required equation.

Note. The value of k in method I or m in method II is found from the second condition satisfied by the line.

Example 1. What is the equation of the straight line joining the origin to the point of intersection of $4x+3y=8$ and $x+y=1$? [*P. U.*]

Method I. [**K-method.**] The equations of the lines are

$$4x+3y-8=0 \quad \ldots (1)$$

$$x+y-1=0 \quad \ldots (2) \quad [\ \text{R. H. S.'s zero (}\textit{Note this step}\text{)}\ ]$$

(*i*) The equation of *any* line thro' the pt. of intersection of the lines (1) and (2) is

$$4x+3y-8+k(x+y-1)=0 \quad \ldots (3) \qquad [\ \text{one line}+k(\text{other line})=0\ ]$$

(*ii*) If it passes thro' (0, 0), then

$$4(0)+3(0)-8+k(0+0-1)=0 \quad \text{or} \quad -8-k=0, \quad \therefore \quad k=-8.$$

(*iii*) Substituting this value of k in (3),

$$4x+3y-8-8(x+y-1)=0$$

or $\quad -4x-5y=0$ or $4x+5y=0$, which is the required equation.

Method II. [**Method of intersection.**] The equations of the lines are

$$4x+3y-8=0 \quad \ldots (1)$$

$$x+y-1=0 \quad \ldots (2)$$

(*i*) Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\begin{matrix} \overset{x}{\overbrace{}} & \overset{y}{\overbrace{}} & \overset{1}{\overbrace{}} \\ 3 \times & -8 \times & 4 \times \\ 1 & -1 & 1 \end{matrix} \quad \begin{matrix} 3 \\ \times \\ 1 \end{matrix}$$ [Art. 23, Rule]

$$\therefore \quad \frac{x}{-3+8}=\frac{y}{-8+4}=\frac{1}{4-3} \quad \text{or} \quad \frac{x}{5}=\frac{y}{-4}=\frac{1}{1}$$

$\therefore$ $x=5$, $y=-4$, which are the co-ordinates of the pt. of intersection.

(*ii*) The equation of the line passing thro' (0, 0) and (5, −4) is

$$\begin{matrix} (1) & (2) \\ (0,\ 0) & (5,\ -4) \end{matrix}$$

$$y-0=\frac{-4-0}{5-0}(x-0) \qquad \left[\ y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\ \right]$$

$$=-\frac{4}{5}x$$

or $5y=-4x$ or $4x+5y=0$, which is the required equation.

Example 2. Find the equation of the line through the intersection of $x-y=1$ and $2x-3y+1=0$, and parallel to $3x+4y=12$. [*Pesh. U. 1955*]

Method I. [**K-method.**] The equations of the lines are

$$x-y-1=0 \quad \ldots (1)$$
$$2x-3y+1=0 \quad \ldots (2) \qquad [\text{ R. H. S.'s zero (}\textit{Note this step}\text{) }]$$

(*i*) The equation of *any* line thro' the pt. of intersection of the lines (1) and (2) is

$$x-y-1+k(2x-3y+1)=0 \quad \ldots (3) \qquad [\text{one line}+k(\text{other line})=0]$$

or $(1+2k)x-(1+3k)y-1+k=0$. $\left[\text{Slope} = -\dfrac{1+2k}{-(1+3k)} = \dfrac{1+2k}{1+3k}\right]$

(*ii*) If it is $\parallel$ to $3x+4y-12=0$ (slope $=-\frac{3}{4}$), then

$$\frac{*1+2k}{1+3k} = -\tfrac{3}{4} \qquad [\ m_1=m_2\ ]$$

or $4+8k = -3-9k$ or $17k=-7$, $\therefore\ k=-\dfrac{7}{17}$.

(*iii*) Substituting this value of k in (3),

$x-y-1-\frac{7}{17}(2x-3y+1)=0$ or $17(x-y-1)-7(2x-3y+1)=0$

or $17x-17y-17-14x+21y-7=0$

or $3x+4y-24=0$, which is the required equation.

Method II. [**Method of intersection.**] The equations of the lines are

$$x-y-1=0 \quad \ldots (1)$$
$$2x-3y+1=0 \quad \ldots (2)$$

(*i*) Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\begin{array}{ccccc} x & & y & & 1 \\ -1 & \times & -1 & \times & 1 \\ -3 & & 1 & & 2 \end{array} \qquad \begin{array}{c} \\ \\ \times \\ \\ \times \end{array} \qquad \begin{array}{c} \\ \\ -1 \\ \\ -3 \end{array}$$

[Art. 23, Rule]

$$\therefore\ \frac{x}{-1-3} = \frac{y}{-2--1} = \frac{1}{-3+2} \quad \text{or} \quad \frac{x}{-4} = \frac{y}{-3} = \frac{1}{-1}$$

$\therefore\ x=\dfrac{-4}{-1}=4,\ y=\dfrac{-3}{-1}=3$, which are the co-ordinates of the pt. of intersection.

(*ii*) The equation of *any* line $\parallel$ to $3x+4y-12=0$ is

$$3x+4y+k=0 \quad \ldots (3) \qquad [\text{ Art. 35, Rule }]$$

If it passes thro' (4, 3), then

$$3(4)+4(3)+k=0 \quad \text{or} \quad 24+k=0, \quad \therefore\ k=-24.$$

Substituting this value of k in (3),

$$3x+4y-24=0, \text{ which is the required equation.}$$

Example 3. Find the equation of the line through the intersection of $y+x=9$ and $2x-3y+7=0$, and perpendicular to the line $2y-3x-5=0$. [*B.H.U.*]

***Or thus :** $\dfrac{1+2k}{3} = \dfrac{-(1+3k)}{4}$

[ratio of the coeffs. of x = ratio of the coeffs. of y (Art. 34-II)]

or $4+8k=-3-9k$ or $17k=-7$, $\therefore\ k=-\dfrac{7}{17}$.

Method I. [**K-method.**] The equations of the lines are

$x+y-9=0$ ... (1)

$2x-3y+7=0$ ... (2) [R.H.S.'s zero (*Note this step*)]

(*i*) The equation of *any* line thro' the pt. of intersection of the lines (1) and (2) is $x+y-9+k(2x-3y+7)=0$ ...(3) [one line$+k$(other line)$=0$]

or $(1+2k)x+(1-3k)y-9+7k=0.$ $\left[\text{Slope} = -\dfrac{1+2k}{1-3k}\right]$

(*ii*) If it is $\perp$ to $2y-3x-5=0$, *i.e.*, $\perp$ to $3x-2y+5=0$

$\left(\text{slope} = -\dfrac{3}{-2} = \dfrac{3}{2}\right)$, then

* $\left(-\dfrac{1+2k}{1-3k}\right)\left(\dfrac{3}{2}\right) = -1$ $[m_1m_2=-1]$

[Cancel (-1)]

or $3+6k=2-6k$ or $12k=-1$, $\therefore$ $k=-\dfrac{1}{12}$.

(*iii*) Substituting this value of k in (3),

$x+y-9-\dfrac{1}{12}(2x-3y+7)=0$ or $12(x+y-9)-(2x-3y+7)=0$

or $12x+12y-108-2x+3y-7=0$

or $10x+15y-115=0$ or, dividing by 5, $2x+3y-23=0$, which is the required equation.

Method II. [**Method of intersection.**] The equations of the lines are

$x+y-9=0$ ... (1)

$2x-3y+7=0$ ... (2)

(*i*) Solving (1) and (2) by cross-multiplication, [Art. 39, Rule]

$$\begin{array}{ccccc} x & & y & & 1 \\ 1 & \times & -9 & 1 & \times \\ -3 & & 7 & 2 & -3 \end{array} \quad \text{[Art. 23, Rule]}$$

$\therefore$ $\dfrac{x}{7-27}=\dfrac{y}{-18-7}=\dfrac{1}{-3-2}$ or $\dfrac{x}{-20}=\dfrac{y}{-25}=\dfrac{1}{-5}$

$\therefore$ $x=\dfrac{-20}{-5}=4$, $y=\dfrac{-25}{-5}=5$, which are the co-ordinates of the pt. of intersection.

(*ii*) The equation of *any* line $\perp$ to $2y-3x-5=0$, *i.e.*, $\perp$ to $3x-2y+5=0$ is $2x+3y+k=0$ ... (3) [Art. 37, Rule]

If it passes thro' (4, 5), then

$2(4)+3(5)+k=0$ or $23+k=0$, $\therefore$ $k=-23$.

Substituting this value of k in (3),

$2x+3y-23=0$, which is the required equation.

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* **Or thus :** $(1+2k)3+(1-3k)(-2)=0$

[product of the coeffs. of x+product of the coeffs. of $y=0$ (Art. 36-II)]

or $3+6k-2+6k=0$ or $1+12k=0$, $\therefore$ $k=-\dfrac{1}{12}$.

EXAMPLES

1. Show that for different values of λ, $ax+by+c+\lambda(a'x+b'y+c')=0$ represents a straight line passing through the point of intersection of $ax+by+c=0$ and $a'x+b'y+c'=0$. [*Bar. U. 1953*]

Find the equation of the line joining the point (2, 3) to the intersection of $2x+3y+1=0$ and $3x-4y=5$. [*D.U. 1938*]

****2. Find the line joining the intersection of $ax+by+c=0$ and $a'x+b'y+c'=0$**

(*i*) **to the point (x', y') ;** [*C. U. 1943*]

(*ii*) **to the origin.** [*Bar. U. 1953*]

3. Find the equation of the line through the intersection of $3x+2y=8$ and $5x-11y+1=0$, and parallel to $6x+13y=25$. [*P.U. 1937S*]

4. Find the equation of the line perpendicular to the line $3x-5y+11=0$ and passing through the intersection of the lines $5x-6y=1$, $3x+2y+5=0$. [*D.U. 1941*]

SECTION IV

LENGTH OF THE PERPENDICULAR FROM A GIVEN POINT ON A GIVEN LINE

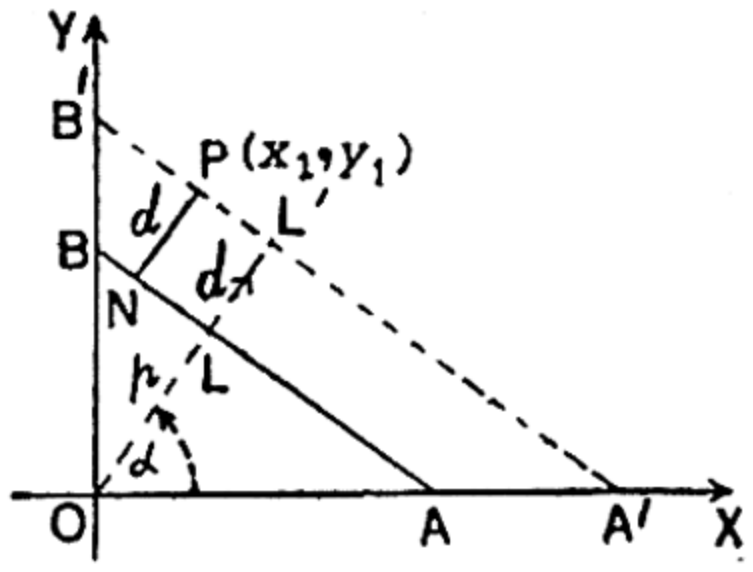
44-I. Perpendicular distance formula. Perpendicular form. To find the perpendicular distance of the point (x_1, y_1) from the line $x \cos \alpha+y \sin \alpha=p$.*

Let AB be the line $x \cos \alpha+y \sin \alpha=p$, and OL the $\perp$ from O upon it, so that $OL=p$ and $\angle XOL=\alpha$.

Let P be the pt. (x_1, y_1) and PN $\perp$ on AB.

Let d be the required $\perp$ distance NP.

Thro' P draw A'B' $\parallel$ to AB to meet OL produced in L'.



Then $OL'=OL+LL'=OL+NP=p+d$, and $\angle XOL'=\alpha$.

(*i*) $\therefore$ the equation of A'B' is
$x \cos \alpha+y \sin \alpha=p+d$. [$\perp$ form (Art. 22)]

(*ii*) $\because$ it passes thro' $P(x_1, y_1)$
$\therefore$ $x_1 \cos \alpha+y_1 \sin \alpha=p+d$.

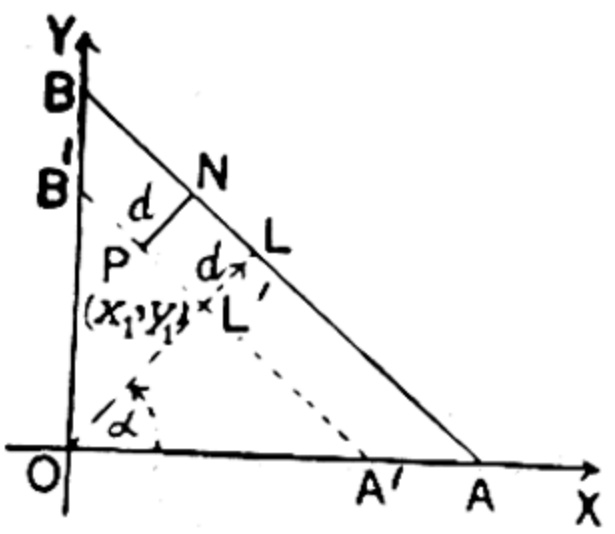
(*iii*) $\therefore$ $\mathbf{d=x_1 \cos \alpha+y_1 \sin \alpha-p}$.

****Note. Complete perpendicular distance formula. Perpendicular form.**

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* [*P.U. 1955*]

If $P(x_1,y_1)$ and the origin (instead of lying on opposite sides of the line AB as in the Fig. of Art. 44-I) lie on the *same* side of AB, as in the adjoining Fig., it may be proved, by proceeding as in Art. 44-I, that



$$d=-(x_1\cos\alpha+y_1\sin\alpha-p)$$

$\therefore$ combining this with the formula of Art. 44-I,

$$\mathbf{d=\pm(x_1\cos\alpha+y_1\sin\alpha-p).}$$

[Rule to find the perpendicular distance of a given point from a given line (equation in the perpendicular form) :

In the L.H.S. of the equation of the line (R.H.S. being zero), substitute the co-ordinates of the pt. The result gives the $\perp$ distance.]

EXAMPLE

Find the length of the perpendicular from the point (h, k) on the line $x\cos\alpha+y\sin\alpha-p=0$. [*P(P). U. 1950S*]

44-II. Perpendicular distance formula. General form. To find the perpendicular distance of the point (x_1, y_1) from the line $ax+by+c=0$*.

The equation of the line is

$$ax+by+c=0.$$ [To reduce it to $\perp$ form]

(*i*) Dividing thro' out by $\sqrt{a^2+b^2}$,

$$\frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y+\frac{c}{\sqrt{a^2+b^2}}=0.$$

(*ii*) Transposing, $\dfrac{a}{\sqrt{a^2+b^2}}x+\dfrac{b}{\sqrt{a^2+b^2}}y=-\dfrac{c}{\sqrt{a^2+b^2}}$,

which is of the $\perp$ form [if c is $-$ve, so that the R. H. S. is $+$ve],

i.e., $\dfrac{a}{\sqrt{a^2+b^2}}x+\dfrac{b}{\sqrt{a^2+b^2}}y+\dfrac{c}{\sqrt{a^2+b^2}}=0$ ... (1) [R. H. S. zero]

$\therefore$ the $\perp$ distance of (x_1, y_1) is

$$\mathbf{d}=\frac{a}{\sqrt{a^2+b^2}}x_1+\frac{b}{\sqrt{a^2+b^2}}y_1+\frac{c}{\sqrt{a^2+b^2}}$$ [Art. 44-I, Rule†]

$$=\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}.$$

Note. Complete perpendicular distance formula. General form. From (1), using the *complete* perpendicular distance formula ($\perp$ form) (Art. 44-I, Note) the $\perp$ distance of (x_1, y_1) is

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*[*P. U. 1956S*]

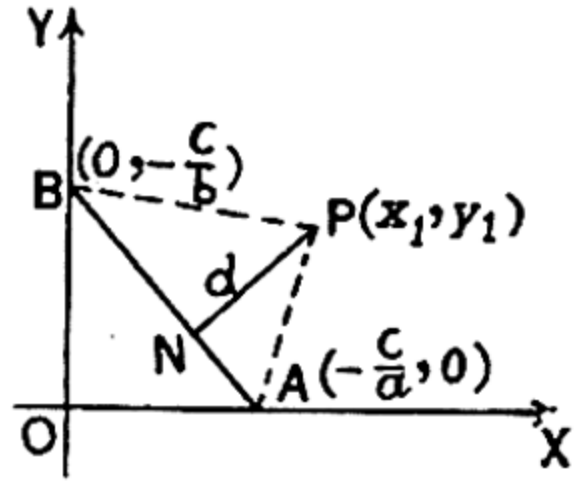
†**Explanation.** In the L. H. S. $\dfrac{a}{\sqrt{a^2+b^2}}x+\dfrac{b}{\sqrt{a^2+b^2}}y+\dfrac{c}{\sqrt{a^2+b^2}}$, substitute the co-ordinates of the pt. (x_1, y_1), thus getting

$$\frac{a}{\sqrt{a^2+b^2}}x_1+\frac{b}{\sqrt{a^2+b^2}}y_1+\frac{c}{\sqrt{a^2+b^2}}.$$

$$\mathbf{d}=\pm\left[\frac{a}{\sqrt{a^2+b^2}}x_1+\frac{b}{\sqrt{a^2+b^2}}y_1+\frac{c}{\sqrt{a^2+b^2}}\right]$$
$$=\pm\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}.$$

Independent method. Let AB be the line $ax+by+c=0$ meeting the axes in A, B, and let P be the pt. (x_1, y_1).

From P draw PN $\perp$ on AB. Let d be the required $\perp$ distance NP. Join PA, PB.



The line $ax+by+c=0$ ... (1)
meets the x-axis where, putting $y=0$ in (1), $ax+c=0$

or $x=-\frac{c}{a}$, $\therefore$ A is $\left(-\frac{c}{a},0\right)$.

Again it meets the y-axis where, putting $x=0$ in (1), $by+c=0$

or $y=-\frac{c}{b}$, $\therefore$ B is $\left(0,-\frac{c}{b}\right)$.

Now $\triangle$ PAB$=\frac{1}{2}$AB.d, $\therefore$ $d=\frac{2\triangle \text{PAB}}{\text{AB}}$ ... (2)

But $\triangle$ PAB$=\frac{1}{2}\left[0+\frac{cy_1}{a}+\frac{c^2}{ab}-0+0+\frac{cx_1}{b}\right]$

$$=\frac{1}{2}\left[\frac{bcy_1+c^2+acx_1}{ab}\right]$$

$$=\frac{c}{2ab}\left[ax_1+by_1+c\right],$$

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ -c/a & 0 \\ 0 & -c/b \\ x_1 & y_1 \end{vmatrix}.$$

and $\text{AB}=\sqrt{\left(0+\frac{c}{a}\right)^2+\left(-\frac{c}{b}-0\right)^2}$

(1) $\left(-\frac{c}{a},0\right)$ (2) $\left(0,-\frac{c}{b}\right)$

$$=\sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}}=\frac{c}{ab}\sqrt{a^2+b^2},$$

$\therefore$ from (2), $d=\dfrac{2.\frac{c}{2ab}\left[ax_1+by_1+c\right]}{\frac{c}{ab}\sqrt{a^2+b^2}}$

$\therefore$ $\mathbf{d}=\dfrac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}$.

Note. Complete perpendicular distance formula. Using the *complete* area formula for the $\triangle$ PAB, [Art. 9, Note 2]

$$\mathbf{d}=\pm\frac{\mathbf{ax_1+by_1+c}}{\sqrt{\mathbf{a^2+b^2}}}.$$

[**Aid to memory.** For the complete perpendicular distance formula (like the complete area formula or the complete angle formula) take the double sign (±) with the ordinary perpendicular distance formula.]

[Rule to find the perpendicular distance of a given point from a given line :

(*i*) *In the L. H. S. of the equation of the line* (*R.H.S. being zero*), *substitute the co-ordinates of the point, and* (*ii*) *divide the result by* $\sqrt{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2}$. *The result gives the perpendicular distance.*

Note. This Rule also gives the result of Art. 44-I and, therefore, covers the Rule of that Art.]

Cor. The perpendicular distance of the origin from the line $ax+by+c=0$ is $\dfrac{c}{\sqrt{a^2+b^2}}$.

For, $d=\dfrac{a(0)+b(0)+c}{\sqrt{a^2+b^2}}$ [Art. 44-II, Rule*]

$=\dfrac{c}{\sqrt{a^2+b^2}}$.

Note. When to use the complete perpendicular distance formula. The complete perpendicular distance formula is used when the length of the perpendicular from the point on the line is *given*. (See Misc. Exs. 37—40, Chap. III.)

Example. Find the distance of the point (−2, 3) from the line x−y=5. [*D. U.*]

The equation of the line is $x-y-5=0$. [R.H.S. zero (*Note this step*)]

$\therefore$ the length of the $\perp$ from $(-2, 3)=\dfrac{(-2)-3-5}{\sqrt{(1)^2+(-1)^2}}$

[Art. 44-II. Rule†]

$=\dfrac{-10}{\sqrt{2}}=-5\sqrt{2}$.

$\therefore$ *changing the sign,* $\perp$ distance *in magnitude* $=5\sqrt{2}$.

EXAMPLES

1. (*a*) **Find the length of the perpendicular from the point** (α,β) **on the line** $ax+by+c=0$. [*A. U. 1945*]

(*b*) Find the length of the perpendicular drawn from

***Explanation.** In the L.H.S. $ax+by+c$, substitute the co-ordinates of the pt. (0, 0), thus getting $a(0)+b(0)+c$, and divide the result by $\sqrt{a^2+b^2}$ [$\sqrt{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2}$]

thus getting $\dfrac{a(0)+b(0)+c}{\sqrt{a^2+b^2}}$.

†**Explanation.** In the L.H.S. $x-y-5$, substitute the co-ordinates of the pt. $(-2, 3)$, thus getting $(-2)-3-5$, and divide the result by $\sqrt{(1)^2+(-1)^2}$ [$\sqrt{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2}$]

thus getting $\dfrac{(-2)-3-5}{\sqrt{(1)^2+(-1)^2}}$.

(*i*) the point (2, −1) upon the straight line $3x-4y=5$.

(*ii*) the point (b, a) upon the straight line $\frac{x}{a}-\frac{y}{b}=1$.

2. (*a*) How far is the line $h(x+h)+k(y+k)=0$ from the origin ? [*C. U.*]

(*b*) If p is the perpendicular distance of the origin from a line whose intercepts on the axes are a and b, show that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$. [*J. & K. U. 1952*]

3. If p and p′ be the perpendiculars from the origin upon the straight lines whose equations are x sec θ+y cosec θ=a and x cos θ −y sin θ=a cos 2θ, prove that $4p^2+p'^2=a^2$. [*A. B. 1951*]

4. Find the length of the perpendicular from (b, a) on the line a(x−a)=b(b−y). [*P. U.*]

5. Find the length of the perpendicular and the co-ordinates of foot of the perpendicular from the point (3, 4) to the straight line $8x+15y+1=0$. [*P. U. 1946 S*]

6. Lengths of altitudes. Find the lengths of the perpendiculars from the vertices on the opposite sides of the triangle whose vertices are the points (0, 0), (1, −1), (3, 2). [*P.U. 1939*]

7. Distance between two parallel lines. Find the perpendicular distance between the lines 9x+40y−20=0 and 9x+40y+21=0. [*D. U.*]

[**Sol. [(⊥) distance between two ∥ lines =length of ⊥ from any pt. on the first line upon the second line.**]

(*i*) In the equation $9x+40y-20=0$, put $x=0$, then $40y-20=0$ or $y=\frac{1}{2}$.

$\therefore$ $(0, \frac{1}{2})$ is a pt. on the first line.

(*ii*) Length of the ⊥ from $(0, \frac{1}{2})$ on the second line $(9x+40y+21=0)$,

$$=\frac{9(0)+40(\frac{1}{2})+21}{\sqrt{(9)^2+(40)^2}}=\frac{20+21}{41}=1,$$ which is the required ⊥ distance.]

8. Find the perpendicular distance between the lines :

(*i*) **3x+4y+15=0 and 3x+4y−9=0.** [*P. U.*]

(*ii*) **3x+4y−5=0 and 6x+8y−45=0.** [*P. U. 1947*]

9. Determine the distance between the lines

(*i*) $3x+4y+5=0$ and $3x+4y+17=0$. [*D. U. 1945*]

(*ii*) $y=2x-4$ and $y=2x+3$. [*P. U.*]

10. Find the distance between the parallel lines :

(*i*) **ax+by+c=0 and ax+by+c′=0.** [*P.U.*]

(*ii*) **y=mx+c and y=mx+d.** [*P. U. 1944*]

**SECTION V

A LINE AND A POINT

Two sides of a line

45. Test for a point to lie on one side or the other of a line. *To show that the quantity ax_1+by_1+c is positive or negative according as the point (x_1, y_1) is on one side or the other of the line $ax+by+c=0$.*

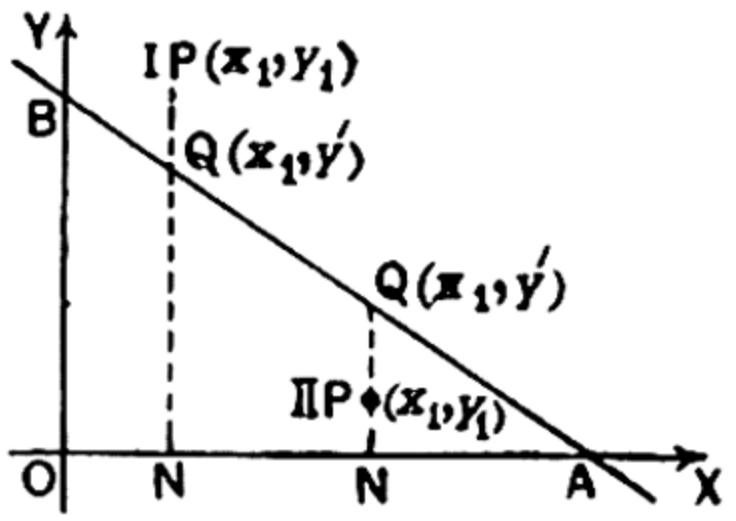
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The equation of the line is

$$ax+by+c=0.$$

Let P be the pt. (x_1, y_1).

From P draw PN $\perp$ on the x-axis to meet the line in Q (x_1, y').



$\because$ Q lies on the line

$\therefore$ $ax_1+by'+c=0$ ... (1)

Now $ax_1+by_1+c=ax_1+by_1+c-0$

[Substitute for 0 from (1)]

$=ax_1+by_1+c-(ax_1+by'+c)=b(y_1-y')=b(\text{NP}-\text{NQ})$ ... (2)

(*i*) If P is on one side of the line as in Fig. I, NP > NQ

$\therefore$ from (2)*, ax_1+by_1+c is +ve. [If b is +ve]

(*ii*) If P is on the other side as in Fig. II, NP < NQ

$\therefore$ from (2), ax_1+by_1+c is $-$ve.

[**Rule to find whether two given points are on the same or on opposite sides of a line :

In the L.H.S. of the equation of the line (R.H.S. being zero), substitute in succession the co-ordinates of the two points ; if the results are of the same sign, the points are on the same side of the line ; if the results are of opposite signs, the points are on opposite sides.]

**EXAMPLES

1. **Show that (2, -1) and (1, 1) are on opposite sides of $3x+4y=6$. [*D. U. 1937*]

[**Sol.** The equation of the line is $3x+4y-6=0$ ...(1)

[R.H.S. zero (*Note this step*)]

In the L.H.S. of (1), substituting the co-ordinates of (2, -1), we get

$$3(2)+4(-1)-6=-4,$$

and substituting the co-ordinates of (1, 1), we get

$$3(1)+4(1)-6=+1 ;$$

$\therefore$ the results are of *opposite* signs,

$\therefore$ the pts. are on *opposite* sides of the line.]

2. **Are the points $(-1, 1)$ and $(1, -1)$ on the same or on opposite sides of the line $x-y=1$?

3. (*a*) The point (x_1, y_1) and the origin are (*i*) on the same side of the line $Ax+By+C=0$, if Ax_1+By_1+C and C have the same sign and (*ii*) on opposite sides of the line, if Ax_1+By_1+C and C have opposite signs.

(*b*) Show that the point $(-3, 4)$ is on the *origin side* of the line $2x-3y=1$.

4. **The sides of a triangle are given by the equations $3x+4y=10$, $4x-3y=5$, and $7x+y+10=0$, show that the origin lies within the triangle. [*B.U.*]

SECTION VI

BISECTORS OF ANGLES BETWEEN TWO STRAIGHT LINES

46. To find the equations of the bisectors of the angles between the straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$.†

---

*$\because$ R.H.S. is +ve, $\therefore$ L.H.S. is also +ve.

†[*J. & K. U. 1957*]

The equations of the lines are $a_1x+b_1y+c_1=0$ ...(1)
$a_2x+b_2y+c_2=0$ ...(2)

(*i*) Let (x, y) be *any* pt. on either of the bisectors.

(*ii*) Then length of the $\perp$ from (x, y) on the line (1) = length of the $\perp$ from (x, y) on the line (2).

(*iii*) $\therefore \dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$, [Art. 44-II, Note]

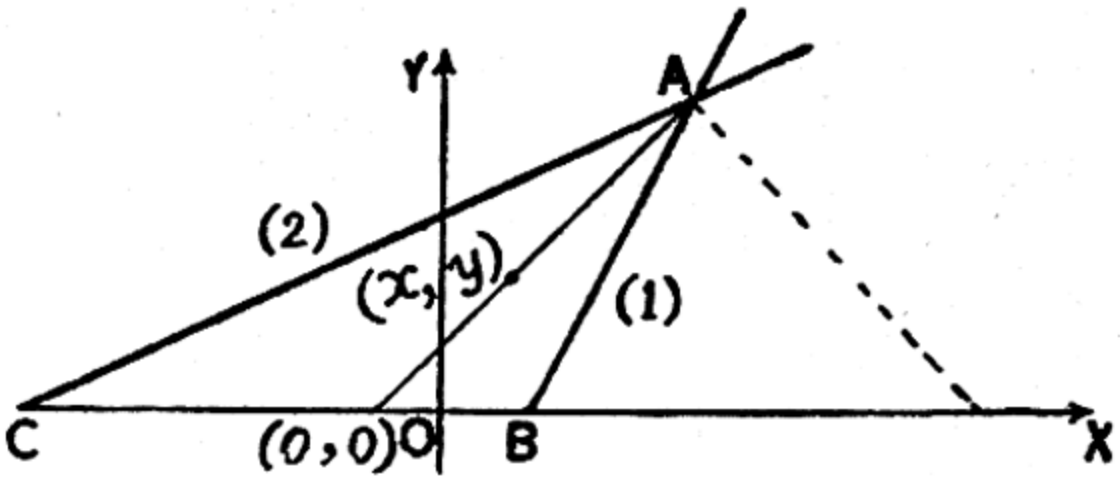
which are the required equations of the bisectors.

Note 1. *To find the equations of the bisectors of the angles between two given lines use the* **complete** *perpendicular distance formula.*

****Note 2. To distinguish between the two bisectors.** *To find the equation of the bisector of the angle in which the origin lies.*

Writing the equations of the lines so that the constant terms are both +ve, let the equations be

$a_1x+b_1y+c_1=0$ ...(1)
$a_2x+b_2y+c_2=0$ ...(2)
[c_1, c_2 *being both* +*ve*]



Let (x, y) be *any* pt. on the bisector of the angle between the lines (1) and (2), in which the origin lies.

Then (x, y) and $(0, 0)$ are on the same side of (1)

$\therefore a_1x+b_1y+c_1$ and [$a_1(0)+b_1(0)+c_1$, *i.e.*,] c_1 are of the same sign (Art. 45). But c_1 is +ve, $\therefore a_1x+b_1y+c_1$ is also +ve.

$\therefore$ length of the $\perp$ from (x, y) on the line $(1) = \dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$.

Similarly length of the $\perp$ from (x, y) on the line $(2) = \dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$.

$\therefore \dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$, which is the equation of the bisector of the angle in which the origin lies.

Similarly the equation of the bisector of the angle in which the origin does not lie, is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

EXAMPLES

1. (*a*) Find the equations of the bisectors of the angles between the straight lines $ax+by+c=0$ and $a_1x+b_1y+c_1=0$.

**State how you will distinguish between them. [*B.U. 1945*]

(*b*) **Find the equations of the bisectors of the angles between the lines $4x-3y+1=0$ and $12x-5y+7=0$.** [*J. & K. U.*]

***Note.** It is superfluous to take $\pm$ sign with *both* sides of the equation; for, if $\pm A=\pm B$, then either (*i*) $A=\pm B$ or (*ii*) $-A=\pm B$, $\therefore A=\mp B$, *i.e.*, $A=\pm B$, which is the same as (*i*).

[**Sol.** (*b*) The equations of the lines are $4x-3y+1=0$ ...(1)
$12x-5y+7=0$ ...(2)

Let (x, y) be *any* pt. on either of the bisectors.
Then length of the $\perp$ from (x, y) on line (1)
= length of the $\perp$ from (x, y) on the line (2)

$$\therefore \quad \frac{4x-3y+1}{5} = \pm \frac{12x-5y+7}{13} \quad \ldots(3)$$

(*i*) Taking the +ve sign in (3),
$52x-39y+13=60x-25y+35$
or $8x+14y+22=0$ or $4x+7y+11=0$.

(*ii*) Taking the −ve sign in (3),
$52x-39y+13=-60x+25y-35$
or $112x-64y+48=0$ or $7x-4y+3=0$
$\therefore$ $4x+7y+11=0$ and $7x-4y+3=0$,
which are the required equations of the bisectors.

[**Check.** The bisectors are $\perp$ to each other.

$$(\because \ m_1m_2 = -\frac{4}{7} \cdot \frac{7}{4} = -1) \ ]$$

****Note.** *To distinguish between the two bisectors.*
$\because$ the constant terms +1 and +7 in (1) and (2) are both +ve,
$\therefore$ taking the +ve sign in (3),
$4x+7y+11=0$, which is the equation of the bisector of the angle in which the origin lies, and the equation of the bisector of the angle in which the origin does not lie, is $7x-4y+3=0$.

2. Perpendicular forms. Find the equations of the bisectors of the angles between the lines $x\cos\alpha+y\sin\alpha=p$ and $x\cos\beta+y\sin\beta=p'$, and show that the two bisectors are perpendicular to each other.
[*P.U. 1942*]

[**Hint.** (*ii*) It will be found that the equations of the bisectors are
$x\cos\alpha+y\sin\alpha-p=\pm(x\cos\beta+y\sin\beta-p')$
i.e., $x(\cos\alpha-\cos\beta)+y(\sin\alpha-\sin\beta)-(p-p')=0$ ...(1)
and $x(\cos\alpha+\cos\beta)+y(\sin\alpha+\sin\beta)-(p+p')=0$ ...(2)

*From (1), $m_1=-\dfrac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}$, $\left[-\dfrac{\text{coeff. of } x}{\text{coeff. of } y} \text{ (Art. 25, Cor.)}\right]$

from (2), $m_2=-\dfrac{\cos\alpha+\cos\beta}{\sin\alpha+\sin\beta}$.

$$\therefore \quad m_1m_2=\frac{\cos^2\alpha-\cos^2\beta}{\sin^2\alpha-\sin^2\beta}=\frac{(1-\sin^2\alpha)-(1-\sin^2\beta)}{\sin^2\alpha-\sin^2\beta}$$
$$=\frac{-(\sin^2\alpha-\sin^2\beta)}{\sin^2\alpha-\sin^2\beta}=-1, \quad \therefore \text{ the bisectors are } \perp.]$$

****3. Find the equations of the bisectors of the angles between $4x+3y-4=0$ and $12x+5y-3=0$. Show that these bisectors are at right angles to each other.** [*P(P). U. 1954*]

***Or thus :** From (1) and (2),
product of the coeffs. of x + product of the coeffs. of y
$=(\cos\alpha-\cos\beta)(\cos\alpha+\cos\beta)+(\sin\alpha-\sin\beta)(\sin\alpha+\sin\beta)$
$=\cos^2\alpha-\cos^2\beta+\sin^2\alpha-\sin^2\beta=(\cos^2\alpha+\sin^2\alpha)-(\cos^2\beta+\sin^2\beta)$
$=1-1=0$, $\therefore$ the bisectors are $\perp$ (Art. 36-*II*).

4. Find the locus of a point which moves so that the perpendiculars drawn from it to the two straight lines $3x+4y=5$, $12x-5y=13$ are equal. [*P. U.*]

****5.** Find the equations to the bisectors of the angles between the straight lines $3x-4y+10=0$ and $5x-12y-10=0$, pointing out the bisector of the angle in which the origin lies. [*J. & K. U.*]

****6. Bisector of acute angle. Find the bisector of the acute angle between the lines $3x+4y=11$ and $12x-5y=2$.** [*P. U. 1931*]

[**Hint.** The equations of the lines are $3x+4y-11=0$ ...(1)
$12x-5y-2=0$ ...(2)

It will be found that the equations of the bisectors are

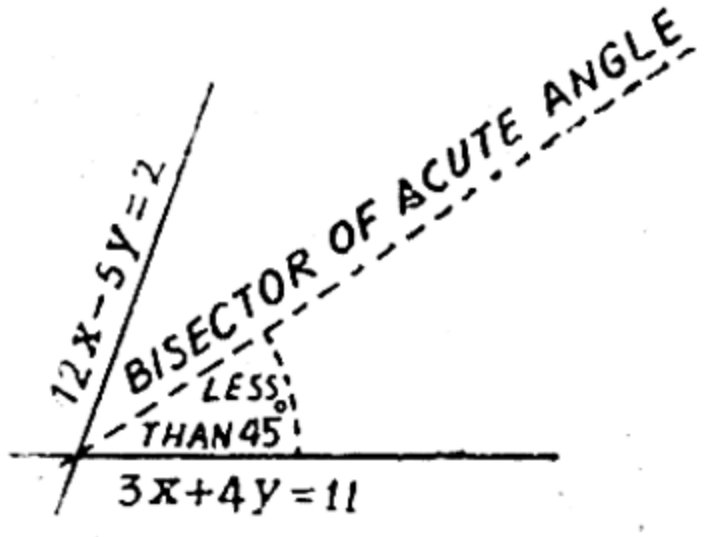
$$\frac{3x+4y-11}{5}=\pm\frac{12x-5y-2}{13} \quad ...(3)$$

(*i*) Taking the +ve sign in (3),
$39x+52y-143=60x-25y-10$
or $21x-77y+133=0$ or, dividing by 7, $3x-11y+19=0$ ...(4)

(*ii*) Taking the −ve sign in (3),
$39x+52y-143=-60x+25y+10$
or $99x+27y-153=0$ or, dividing by 9, $11x+3y-17=0$ ...(5)

The angle between the lines being *acute* ($<90°$), the angle between one of the lines and the bisector of the acute angle is $<\frac{1}{2}.90°$, *i.e.*, $<45°$.

Let θ be the angle between one of the lines (1) (say) and one of the bisectors (4) (say).



Then $\tan\theta=\dfrac{(-\frac{3}{4})-(\frac{3}{11})}{1+(-\frac{3}{4})(\frac{3}{11})}=\dfrac{-\frac{45}{44}}{\frac{35}{44}}=-\frac{9}{7}$,

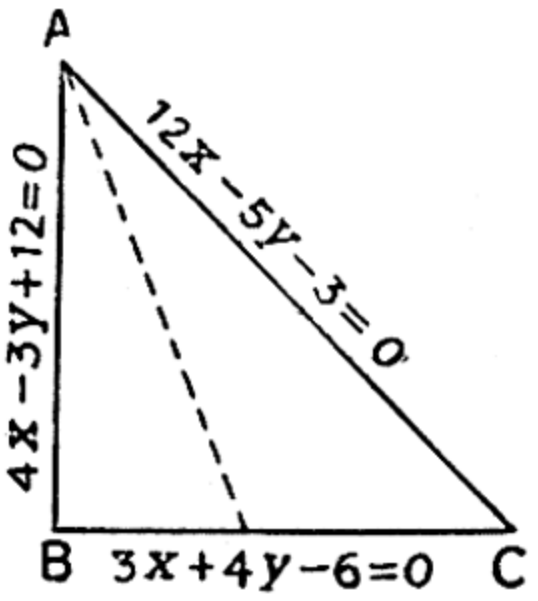
which is *numerically* >1, $\therefore$ *angle* $>45°$.

$\therefore$ (4) is *not* the bisector of the acute angle but the *other* bisector (5).
i.e., $11x+3y-17=0$ is the bisector of the acute angle.]

****7. Find the bisector of the acute angle between the lines $5x=12y+24$ and $12x=5y+10$.** [*P. U.*]

****8. Internal bisector.** A triangle is formed by the lines whose equations are $3x+4y-6=0$, $12x-5y-3=0$ and $4x-3y+12=0$. Find the internal bisector of the vertical angle opposite to the side $3x+4y-6=0$. [*P. U. 1933*]

[**Sol.** Let the three lines be BC, CA, AB. The bisectors of the vertical angle opposite to $3x+4y-6=0$, *i.e.*, the angle between the lines $12x-5y-3=0$ and $4x-3y+12=0$, are



$$\frac{12x-5y-3}{13}=\pm\frac{4x-3y+12}{5}$$

i.e., $8x+14y-171=0$ ...(1)
and $112x-64y+141=0$ ...(2)

Solving $4x-3y+12=0$ and $3x+4y-6=0$, the pt. B is $\left(-\frac{6}{5}, \frac{12}{5}\right)$.

Solving $3x+4y-6=0$ and $12x-5y-3=0$, the pt. C is $(\frac{2}{3}, 1)$.

The pts. B and C are on opposite sides of the internal bisector (shown dotted in the Fig.).

In the L.H.S. of the bisector (1), substituting the co-ordinates of $\left(-\frac{6}{5}, \frac{12}{5}\right)$, we get $8\left(-\frac{6}{5}\right)+14\left[\frac{12}{5}\right]-171=-147$,

and substituting the co-ordinates of $(\frac{2}{3}, 1)$, $8(\frac{2}{3})+14(1)-171=-\frac{455}{3}$;

$\therefore$ the results are of the *same* sign,

$\therefore$ B, C are on the *same* side of (1).

$\therefore$ (1) is *not* the internal bisector, but the *other* bisector (2),

i.e., $112x-64y+141=0$ is the internal bisector.]

****9.** A triangle is formed by joining the points (2, −2), (8, −2), (8, 6) ; find the equation of the internal bisector of the angle at the vertex (8, 6). [*D.U. 1939*]

MISCELLANEOUS EXAMPLES ON CHAPTER III

1. *A straight line moves so that the sum of the reciprocals of its intercepts on a pair of intersecting (perpendicular) straight lines is constant : show that it passes through a fixed point.* [*C.U. 1945*]

[**Sol.** Let the intercepts be a and b. Then the equation of the line is $\frac{x}{a}+\frac{y}{b}=1$ ...(1) Also $\frac{1}{a}+\frac{1}{b}=$constant (Given) $=\frac{1}{c}$ (say),

$\therefore \quad \frac{c}{a}+\frac{c}{b}=1, \quad \therefore$ (1) passes thro' the fixed pt. (c, c).]

2. OA and OB are two perpendicular straight lines. The straight line AB is drawn in such a manner that OA+OB=8. Find the locus of the middle point of AB. [*P.U.*]

3. *A line is drawn through a fixed point (h, k) cutting the axes in P and Q respectively, and the rectangle OPRQ is completed. Find the equation of the locus of R.* [*P.U. 1934S*]

4. Prove that the straight line $ax+by+c=0$ divides the join of (x_1, y_1) and (x_2, y_2) in the ratio $-\frac{ax_1+by_1+c}{ax_2+by_2+c}$. [*B.U. 1934*]

5. Reduce the equation $\sqrt{3}x-y=2$ to the tangent form, $x-2y=3$ to the intercept form and $x-\sqrt{3}y=4$ to the perpendicular form and explain each result geometrically.

6. (*i*) Deduce the form $y=mx+c$ from the form $x\cos\alpha+y\sin\alpha=p$. [*P.U.*]

(*ii*) Deduce the form $\frac{x}{a}+\frac{y}{b}=1$ from the form $x\cos\alpha+y\sin\alpha=p$. [*P.U. 1944S*]

(*iii*) *Deduce the form $x\cos\alpha+y\sin\alpha=p$ from the form* $\frac{x}{a}+\frac{y}{b}=1$. [*P.U. 1943S*]

[**Hints.** Take the Fig. of Art. 22. $\angle OBL=90°-\angle LOB=\angle AOL=\alpha$.

(*i*) Reduce $x\cos\alpha+y\sin\alpha=p$ to the tangent form.

Then $y=-x\cot\alpha+p\operatorname{cosec}\alpha$ ...(1) Now let $\angle XAB=\theta$. Then from $\triangle$ OLA, $\alpha=\theta-90°$ and from $\triangle$ OLB, $p\operatorname{cosec}\alpha=OB=c$. Substitute in (1). Put $\tan\theta=m$.

(*ii*) Reduce $x\cos\alpha+y\sin\alpha=p$ to the intercept form. Then

$$\frac{x}{p\sec\alpha}+\frac{y}{p\operatorname{cosec}\alpha}=1 \quad ...(2)$$

Now from $\triangle$ OLA, $p\sec\alpha=OA=a$,
and from $\triangle$ OLB, $p\operatorname{cosec}\alpha=OB=b$. Substitute in (2).

(*iii*) $a=OA$. $\left[\text{But } \frac{OA}{OL}=\sec\alpha\right]$ $\therefore$ $a=p\sec\alpha$. Again $b=OB$.

$\left[\text{But } \frac{OB}{OL}=\operatorname{cosec} OBL=\operatorname{cosec}\alpha\right]$ $\therefore$ $b=p\operatorname{cosec}\alpha$. Substitute these values of a and b in $\frac{x}{a}+\frac{y}{b}=1$, and multiply the resulting equation by p.]

7. Find the equation of the straight line which bisects the distance between the points (a, b) and (a', b') and also bisects the distance between the points $(a, -b)$ and $(-a', b')$.

8. Prove that the three points $(-1, -1)$, $(5, 7)$ and $(8, 11)$ lie in a straight line. Find the intercepts which it makes on the axes. [*P.U. 1945*]

9. *Find the length of the straight line drawn through a given point (x_1, y_1) in a given direction (i.e., making an angle θ with the x-axis) to meet a given line $(Ax+By+C=0)$.* [*C.U. 1954*]

[**Hint.** The equation of the line is $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$. *Any* pt. on the line is $(x_1+r\cos\theta, y_1+r\sin\theta)$. If it lies on $Ax+By+C=0$, find the value of r.]

****10. Case of failure of angle formula I.** Prove that $(1, -2)$, $(2, 0)$, $(1, 6)$, and $(0, 4)$ are the co-ordinates of the angular points of a parallelogram, and find the angle between the diagonals.

[**Hint.** (*b*) Let A(1, −2), B(2, 0), C(1, 6), D(0, 4) be the pts. Then,

$$m_1=\text{slope of AC}=\frac{6-(-2)}{1-1}=\frac{8}{0}=\infty,$$

$\therefore$ *the angle formula :* $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ *fails.* (See Art. 33-II, Note.)

It will be found that the equation of AC is $x-1=0$ ...(1)
[Form $Ax+By+C=0$]

and that of BD is $2x+y-4=0$ ...(2)

Compare (1) with $a_1x+b_1y+c_1=0$ and (2) with $a_2x+b_2y+c_2=0$.

Here $a_1=1$, $b_1=0$; $a_2=2$, $b_2=1$.

$\therefore$ if θ is the angle between the diagonals, then

$$\tan\theta=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}$$

$$=\frac{2(0)-1(1)}{1(2)+0(1)}=-\tfrac{1}{2}.\ ]$$

**11. Prove that the equation to the straight line which passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and is perpendicular to $x\sec\theta+y\operatorname{cosec}\theta=a$ is $x\cos\theta-y\sin\theta=a\cos 2\theta$.

12. Find the equations to the straight lines which divide, internally and externally, the line joining $(-1, -3)$ to $(5, 3)$ in the ratio $1:2$ and which are perpendicular to this line.

13. Find the equations of the straight lines which pass through the point (h, k) and are inclined at an angle $\tan^{-1} m$ to the straight line $y=mx+c$.

14. (1, 2) and (3, 8) are a pair of opposite vertices of a square. Find the equations of the sides and diagonals of the square.

[*B.U. 1943*]

15. Find the point of intersection of the lines :

(*i*) $y=m_1x+\dfrac{a}{m_1}$ and $y=m_2x+\dfrac{a}{m_2}$;

(*ii*) $x\cos\theta_1+y\sin\theta_1=a$ and $x\cos\theta_2+y\sin\theta_2=a$.

16. Find the point of intersection and the inclination of the two straight lines $ax-by=a-b$ and $a(x+y)+b(x-y)=2a$.

17. Find the co-ordinates of the point in which the line $2x-3y+4=0$ meets the line joining the two points $(-1, 1)$ and $(-3, 0)$, and determine also the angle at which they cut one another.

18. Find the equations to the diagonals of the rectangle the equations of whose sides are $x=a$, $x=a'$, $y=b$ and $y=b'$. [*C. U. 1951*]

19. (*a*) *If* $u=lx+my+n=0$ *and* $v=l'x+m'y+n'=0$, *interpret the equation* $u+kv=0$. [*B. U. 1946*]

(*b*) *Find the equation of the straight line passing through the intersection of the straight lines* $3x-4y+1=0$ *and* $5x+y-1=0$ *and cutting off equal intercepts from the co-ordinate axes.* [*N.U. 1951*]

[****Note. Important.** *If a line cuts off equal intercepts from the axes, its equation is* $\dfrac{x}{a}+\dfrac{y}{a}=1$ *or* $x+y=a$, $\therefore$ *its slope* $=-1$.]

20. Find the equations of the straight lines which pass through the origin and trisect the portion of the straight line $2x+y=6$, which is intercepted between the axes.

21. Find the equation of the straight line drawn at right angles to the line $\dfrac{x}{a}-\dfrac{y}{b}=1$ through the point where it meets the axis of x.

[*P.U. 1942S*]

22. Find the area of the triangle formed by the straight lines $y=ax-bc$, $y=bx-ca$, $y=cx-ab$.

23. *Find the locus of the foot of the perpendicular from the origin to the line which always passes through a fixed point* (h, k). [*P.U. 1935*]

[**Hint.** The equation of *any* line thro' (h, k) is $y-k=m(x-h)$ ...(1) The equation of the line $\perp$ to it and passing thro' the origin is

$y=-\frac{1}{m}x$...(2) To find the locus of (the ft. of the $\perp$, *i.e.*,) the pt. of intersection of (1) and (2), *eliminate m from (1) and (2)* (by multiplying (1) and (2)).]

[**Rule to find the locus of the point of intersection of two lines whose equations involve one variable :**

Eliminate the variable from the equations of the two lines.]

24. If three straight lines $p_1x+q_1y=1$, $p_2x+q_2y=1$ and $p_3x+q_3y=1$ all pass through one point, show that the three points (p_1, q_1), (p_2, q_2) and (p_3, q_3) lie in one straight line. [*P(P). U. 1953*]

25. Find the equation of the straight line passing through the origin and the point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=1 \text{ and } \frac{x}{b}+\frac{y}{a}=1.$$

26. Orthocentre. A triangle is determined by the lines $y+x-6=0$, $3y-x+2=0$, $3y=5x+2$. Find the co-ordinates of its orthocentre. [*P.U. 1955*]

27. *Obtain the equation of the line through the intersection of $2x-3y+4=0$ and $3x+4y-5=0$ parallel to the axis of y.* [*P(P). U. 1955*]

28. *Find the equation of the line passing through the intersection of $x+2y+1=0$ and $2x-3y=5$, and the intersection of the lines $3x-4y=11$ and $4x+5y+6=0$.*

29. *A straight line is such that the sum of the perpendiculars upon it from any number of fixed points is zero, show that it always passes through a fixed point.*

[**Sol.** Let the equation of the line be $x\cos\alpha+y\sin\alpha-p=0$ ...(1) and let the fixed pts. be (x_1, y_1), (x_2, y_2),..., (x_n, y_n). Then from the given condition

$x_1\cos\alpha+y_1\sin\alpha-p+x_2\cos\alpha+y_2\sin\alpha-p+\ldots+x_n\cos\alpha+y_n\sin\alpha-p=0$,

$\therefore (x_1+x_2+\ldots+x_n)\cos\alpha+(y_1+y_2+\ldots+y_n)\sin\alpha-np=0.$

Dividing thro' out by n,

$$\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\cos\alpha+\left(\frac{y_1+y_2+\ldots+y_n}{n}\right)\sin\alpha-p=0,$$

$\therefore$ (1) passes thro' the fixed pt. $\left(\frac{x_1+x_2+\ldots+x_n}{n}, \frac{y_1+y_2+\ldots+y_n}{n}\right)$.]

[**Note. Important. For problems relating to lengths of perpendiculars on lines, take the equations of the lines** *in the perpendicular form.*]

30. *Show that the locus of a point which moves so that the sum of the perpendiculars let fall from it upon two given straight lines is constant is a straight line.* [*M.U. 1936*]

31. Find the distance of the point of intersection of the two straight lines $x+2y-3=0$, and $2x+y=0$, from the straight line $3x-4y=4$.

32. *Find the equation of the line drawn through the origin perpendicular to the line $3x-4y=5$; find also the length of the perpendicular.*

33. Find the perpendicular distance from the origin of the perpendicular from the point (1, 2) upon the straight line $\sqrt{3}y=x+4$.
[*P.U. 1943*]

34. **Lengths of altitudes. The equations of the sides of a triangle are $3x+y+4=0$, $3x-5y+34=0$, $3x-2y+1=0$. Find the lengths of the altitudes.** [*D.U. 1933*]

35. Write down the equations of two straight lines through the points (1, 1) and (−2, −1) parallel to the line $3x+4y+7=0$, and find the perpendicular distance between them. [*P. U. 1942*]

36. **Find the centre and radius of the circle which is inscribed in the triangle formed by the straight lines**
$$y=0,\ 12x-5y=0, \text{ and } 3x+4y-7=0.$$

37. What are the points on the x-axis whose perpendicular distance from the line $\frac{x}{a}+\frac{y}{b}=1$ is a? [*C. U. 1951*]

[**Note. Important.** *When the length of the perpendicular is given, use the complete ⊥ distance formula.*]

38. **Find the points on the straight line $3x-2y=2$ at a distance of 3 units from the straight line $3x+4y=8$.** [*B.U. 1942*]

39. Find the equations of two straight lines which are parallel to the straight line $x+7y+2=0$, and at a unit distance from the point $(2, -1)$. [*P.U. 1942S*]

40. Find the equations of the two straight lines drawn through the point (0, 1) on which the perpendiculars dropped from the point (2, 2) are each of unit length. [*P.U.*]

41. **Find the equations of the straight lines bisecting the angles between the straight lines $4y+3x-12=0$ and $3y+4x-24=0$ and draw the four lines to scale in one diagram on a squared paper.** [*M.U.*]

42. Find the equation of a straight line through the origin and through the intersection of the straight lines $x-y-4=0$, $7x+y+20=0$, and prove that it bisects the angle between them. [*Pesh. U. 1951*]

[**Hint.** (*ii*) Find the bisectors of the angles between the given lines, and show that one of them is the line found in part (*i*) of the question.]

43. **Show that the perpendiculars let fall from any point of the straight line $7x+4y=3$ upon the two straight lines $3x-4y=2$ and $5x-12y=4$ are equal to each other.** [*J. & K. U.*]

[**Hint.** Let (x_1, y_1) be a pt. the ⊥s from which on the two lines are equal. Then it will be found that $7x_1+4y_1-3=0$, $\therefore$ (x_1, y_1) lies on the line $7x+4y-3=0$, *i.e.*, on the line $7x+4y=3$.]

**44. Show that the origin is within the triangle whose angular points are (1, 2), (−2, 3) and (−1, −4).

CHAPTER IV

[☞This chapter should be omitted by the Intermediate students of the Punjab University.]

PAIR OF STRAIGHT LINES REPRESENTED BY A QUADRATIC EQUATION

SECTION I

EQUATION OF A PAIR OF STRAIGHT LINES

47. *To find the locus of the equation*

$$(ax+by+c)(a'x+b'y+c')=0.$$

The equation is $(ax+by+c)(a'x+b'y+c')=0$ ...(1)

The co-ordinates of the pts. which satisfy (1)

satisfy *either* $ax+by+c=0$...(2)

or $a'x+b'y+c'=0$...(3)

[$\because$ if $AB=0$, then either $A=0$ or $B=0$]

$\therefore$ the pts. on the locus of (1) lie either on the st. line (2) (Art. 24) or on the st. line (3),

i.e., the locus of (1) consists of the two st. lines (2) and (3).

$\therefore$ $(ax+by+c)(a'x+b'y+c')=0$ represents the two st. lines $ax+by+c=0$, and $a'x+b'y+c'=0$.

Cor. *If the separate equations of two straight lines are* $ax+by+c=0$, *and* $a'x+b'y+c'=0$, *then their combined equation is*

$$\mathbf{(ax+by+c)(a'x+b'y+c')=0.}$$

[Rule to find the combined equation of two straight lines whose separate equations are given :

(*i*) *Write the equation of each straight line so that the R. H. S. is zero.* **(Note this step)**

(*ii*) *Multiply the L. H. S.'s of the two resulting equations, and equate the product to zero.*]

Example. *What loci are represented by the following equations :*

(*i*) $xy=0$, (*ii*) $x^2-3x+2=0$ *?*

(*i*) The equation is $xy=0$,

which represents the two st. lines $x=0$, and $y=0$ (Art. 47), *i.e.*, the y-axis (Art. 16, Cor.) and x-axis (Art. 15, Cor.).

(*ii*) The equation is $x^2-3x+2=0$ ...(1)

[Method of factors.]

Factorising, $(x-1)(x-2)=0$.

$\therefore$ (1) represents the two st. lines $x-1=0$, and $x-2=0$ (Art. 47), or $x=1$, and $x=2$, which are $\parallel$ to the y-axis and at distances 1 and 2 respectively from it (Art. 16).

EXAMPLES

1. What loci are represented by the equations,
(i) $y^2-4xy+3x^2=0$, (ii) $xy-3x-4y+12=0$?

(*i*) The equation is $y^2-4xy+3x^2=0$ ...(1)

[Method of factors.]

Factorising, $(y-x)(y-3x)=0$.

$\therefore$ (1) represents the two st. lines $y-x=0$, and $y-3x=0$,

or $y=x$...(2), and $y=3x$...(3).

[Comparing (2) with $y=mx$, here $m=1$, *i.e.*, $\tan\theta=1=\tan 45°$, $\therefore\ \theta=45°$. Again, comparing (3) with $y=mx$, here $m=3$, *i.e.*, $\tan\theta=3$, $\therefore\ \theta=\tan^{-1}3$.]

i.e., (1) represents two st. lines passing thro' the origin and making angles of 45° and $\tan^{-1}3$ with the x-axis.

(*ii*) The equation is $xy-3x-4y+12=0$ ...(1)

[Method of factors.]

Factorising, $x(y-3)-4(y-3)=0$

or $(y-3)(x-4)=0$

$\therefore$ (1) represents the two st. lines $y-3=0$, and $x-4=0$,

or $y=3$, and $x=4$,

i.e., (1) represents two st. lines, one $\|$ to the x-axis and at a distance 3 from it, and the other $\|$ to the y-axis and at a distance 4 from it.

Find what loci are represented by the equations (Exs. 2—4) :—

2. (*i*) $x^2-y^2=0$, (*ii*) $x^2-xy=0$.

3. (*i*) $y^2-a^2=0$, (*ii*) $xy-ay=0$.

4. (*i*) $(x+y)^2-a^2=0$, (*ii*) $(x+a)^2-y^2=0$.

5. The equations to a pair of opposite sides of a parallelogram are $x^2-7x+6=0$ and $y^2-14y+40=0$, find the equations to its diagonals. [*A. U. 1931*]

6. Show that the equation $(x-a)(y-b)=0$ represents a pair of straight lines, and find the angle between them.

7. Find the equation of the lines which trisect the angle between the axes (supposed rectangular). [*P. U. B. 1917*]

48. Homogeneous equation. Def. An equation in every term of which the sum of the indices of x and y is the *same*, is called a **homogeneous equation.**

Thus the equation $ax^2+2hxy+by^2=0$ is a **homogeneous equation of the second degree ;** for in the first term the index of $x=2$; in the second term the sum of the indices of x and $y=1+1=2$; and in the third term the index of $y=2$.

49. To prove that a homogeneous equation of the second degree represents two straight lines through the origin.*

Let the homogeneous equation of the second degree be

$$ax^2+2hxy+by^2=0 \quad ...(1)$$

or $$by^2+2hxy+ax^2=0.\dagger$$

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*[*B. H. U. 1947*]

†**Why this step.** $\because$ the equation of a straight line thro' the origin is $y=mx$ or $y-mx=0$, $\therefore$ we write the equation in descending powers of y.

Dividing thro' out by x^2,

$$b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a=0 \quad ...(2)$$

which is a quadratic in $\frac{y}{x}$

Let m_1, m_2 be the roots of this quadratic.

Then $\frac{y}{x}=m_1$, or $\frac{y}{x}=m_2$.

$\therefore$ (1) represents the two st. lines $y=m_1x$, and $y=m_2x$, which pass thro' the origin.

Note. Important. *If in a numerical example, it is required to find what straight lines are represented by the given (homogeneous) equation, (factorise the L.H.S. of the given equation or) solve the given equation as a quadratic in x (or y).* (*See Ex. 1, Art. 50.*)

Cor. 1. Two important results. *If $y=m_1x$, and $y=m_2x$ are the separate equations of the lines $ax^2+2hxy+by^2=0$, then*

$$\mathbf{m_1+m_2=-\frac{2h}{b}, \text{ and } m_1m_2=\frac{a}{b}.}$$

(See Art. 49.) (2) is a quadratic in $\frac{y}{x}$, and m_1, m_2 are the roots.

$\therefore$ $\mathbf{m_1+m_2}$=sum of the roots$=-\frac{\mathbf{2h}}{\mathbf{b}}$,

and $\mathbf{m_1m_2}$=product of the roots$=\frac{\mathbf{a}}{\mathbf{b}}$.

Note. *The two results : $m_1+m_2=-\frac{2h}{b}$, and $m_1m_2=\frac{a}{b}$ are extremely important. They enable us to solve many problems relating to the lines $ax^2+2hxy+by^2=0$ without finding their separate equations which involve radicals and are, therefore, cumbersome to deal with.*

****Cor. 2.** *The lines $ax^2+2hxy+by^2=0$ are real and different, real and coincident, or imaginary according as h^2-ab is positive, zero, or negative.*

(See Art. 49.) The lines (1) are real and different, real and coincident, or imaginary according as the roots of the quadratic (2) are real and different, equal, or imaginary, *i.e.*,

according as $4h^2-4ab$ ["b^2-4ac"]

is +ve, 0, or −ve

or, dividing by 4, (which is +ve)

according as h^2-ab is +ve, 0, or −ve.

Example. Find the condition that one of the straight lines given by the equation $ax^2+2hxy+by^2=0$ may coincide with one of those given by the equation $a'x^2+2h'xy+b'y^2=0$. [*A. B. 1951*]

The equation of the first pair of lines is

$$ax^2+2hxy+by^2=0 \quad ...(1)$$

and that of the second pair is

$$a'x^2+2h'xy+b'y^2=0 \quad ...(2)$$

Let the equation of one line of the pair (1) coincident with one line of the pair (2) be

$$y=mx \quad \ldots(3)$$

Then this value of y satisfies both (1) and (2),

$\therefore$ $ax^2+2hx.mx+bm^2x^2=0,$

and $a'x^2+2h'x.mx+b'm^2x^2=0$ [Cancel x^2]

or $bm^2+2hm+a=0 \quad \ldots(4)$

and $b'm^2+2h'm+a'=0 \quad \ldots(5)$

[*To eliminate m from (4) and (5).*]

By cross-multiplication,

$$\frac{m^2}{2(ha'-h'a)}=\frac{m}{ab'-a'b}=\frac{1}{2(bh'-b'h)}.$$

*From the first and third members,

$$m^2=\frac{2(ha'-h'a)}{2(bh'-b'h)}=\frac{ha'-h'a}{bh'-b'h} \ldots(6)$$

and from the second and third members,

$$m=\frac{ab'-a'b}{2(bh'-b'h)}$$

$\therefore$ $$m^2=\frac{(ab'-a'b)^2}{4(bh'-b'h)^2} \ldots(7)$$

From (6) and (7),

$$\frac{ha'-h'a}{bh'-b'h}=\frac{(ab'-a'b)^2}{4(bh'-b'h)^2}$$ [Cancel $bh'-b'h$ from the denoms.]

or $4(ha'-h'a)(bh'-b'h)=(ab'-a'b)^2,$

which is the required condition.

EXAMPLES

1. Find the condition that one of the lines $ax^2+2hxy+by^2=0$ may be perpendicular to one of the lines $a'x^2+2h'xy+b'y^2=0$. [*P. U. B. 1953S*]

[**Hint.** The equation of the first pair of lines is

$$ax^2+2hxy+by^2=0 \quad \ldots(1)$$

and that of the second pair is

$$a'x^2+2h'xy+b'y^2=0 \quad \ldots(2)$$

Let the equation of one line of the pair (1) be

$$y=mx \quad \ldots(3)$$

Then this value of y satisfies (1). It will be found (as in solved Ex., Art. 49) that

$$bm^2+2hm+a=0 \quad \ldots(4)$$

The equation of the line thro' the origin $\perp$ to (3) is $y=-\frac{1}{m}x.$

This is a line of the pair (2)

$\therefore$ this value of y satisfies (2). It will be found that

$$a'm^2-2h'm+b'=0 \quad \ldots(5)$$

Eliminate m from (4) and (5). (Proceed as in solved Ex., Art. 49.)]

2. Shew that the equation $bx^2-2hxy+ay^2=0$ represents a pair

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*Or thus : **Short-cut.** $\therefore$ *product of the extremes* (*i.e.*, extreme denoms.) = *square of the mean* (*i.e.*, mean denom.)

or $$2(ha'-h'a).2(bh'-b'h)=(ab'-a'b)^2.$$

of straight lines which are at right angles to the pair given by the equation $ax^2+2hxy+by^2=0$. [*J. & K. U. 1955*]

[**Hint.** The equation of the lines is $ax^2+2hxy+by^2=0$ ...(1)
Let the equation of *either* line of the pair (1) be $y=mx$ ...(2). Then this value of y satisfies (1). It will be found that $a+2hm+bm^2=0$ ...(3)

The equation of the line thro' the origin $\perp$ to (2) is $y=-\frac{1}{m}x$ ...(4)

Eliminate m from (3) and (4) [by substituting its value $\left(m=-\frac{x}{y}\right)$ from (4) in (3)].]

3. The distance of a point (x_1, y_1) from each of two straight lines, which pass through the origin of co-ordinates, is δ ; prove that the two lines are given by $(x_1y-xy_1)^2=\delta^2(x^2+y^2)$. [*R. B. 1940*]

4. Prove that the product of the perpendiculars let fall from the point (x', y') upon the pair of straight lines $ax^2+2hxy+by^2=0$ is

$$\frac{ax'^2+2hx'y'+by'^2}{\sqrt{(a-b)^2+4h^2}}.$$ [*B.H.U. 1951*]

[**Hint.** The equation of the lines is $ax^2+2hxy+by^2=0$ ...(1)
Let the separate equations of the lines be
$y=m_1x$ ...(2), and $y=m_2x$ ...(3)

*Then $m_1+m_2=-\frac{2h}{b}$, and $m_1m_2=\frac{a}{b}$ ...(4) (Art. 49, Cor. 1)

The product of the $\perp$s from (x', y') on the lines (2) and (3)

$$=\frac{(y'-m_1x')}{\sqrt{1+m_1^2}}\cdot\frac{(y'-m_2x')}{\sqrt{1+m_2^2}}=\frac{y'^2-(m_1+m_2)x'y'+m_1m_2x'^2}{\sqrt{1+m_1^2+m_2^2+m_1^2m_2^2}}$$

$$=\frac{y'^2-(m_1+m_2)x'y'+m_1m_2x'^2}{\sqrt{1+(m_1+m_2)^2-2m_1m_2+m_1^2m_2^2}} \quad ...(5)$$

Substitute the values of m_1+m_2 and m_1m_2 from (4) in (5).]

5. Shew that the area of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ is

$$\frac{n^2\sqrt{(h^2-ab)}}{am^2-2hlm+bl^2}.$$ [*P.U.B. 1944S*]

SECTION II

ANGLE FORMULA FOR A PAIR OF STRAIGHT LINES THROUGH THE ORIGIN

50. Angle formula for a pair of lines through the origin. To find the angle between the straight lines given by the equation $ax^2+2hxy+by^2=0$†

***Or thus :** Then $ax^2+2hxy+by^2=b(y-m_1x)(y-m_2x)$ (identically)
[Adjusting the coeff. of y^2 on the R.H.S. from that on the L.H.S.]
$=b\,[\,y^2-(m_1+m_2)xy+m_1m_2x^2\,]$.

Equating coeffs. of xy and x^2 on both sides,
$-b(m_1+m_2)=2h$, $bm_1m_2=a$, or $m_1+m_2=-\frac{2h}{b}$, and $m_1m_2=\frac{a}{b}$.

†[*J. & K. U. B. 1954*]

The equation of the lines is $ax^2+2hxy+by^2=0$ ...(1)
Let the separate equations of the lines be
$y=m_1x$ ...(2), and $y=m_2x$ ...(3)

Then $m_1+m_2=-\frac{2h}{b}$, and $m_1m_2=\frac{a}{b}$ ...(4) (Art. 49, Cor. 1)

If θ is the angle between the lines (2) and (3), then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2} \quad \text{(Art. 33-I)}$$

$$=\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2} \quad \text{[Substitute from (4)]}$$

$$=\frac{\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}}{1+\frac{a}{b}}$$

or $\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$, $\therefore\ \theta=\tan^{-1}\frac{2\sqrt{h^2-ab}}{a+b}$.

Cor. 1. Conditions of perpendicularity and coincidence. To find the condition that the straight lines $ax^2+2hxy+by^2=0$ may be (*i*) perpendicular, (*ii*) coincident.*

(*i*) If the lines are $\perp$,
θ, the angle between them, $=90°$
$\therefore\ \tan\theta=\tan 90°=\infty$,

i.e., $\frac{2\sqrt{h^2-ab}}{a+b}=\infty$ (Art. 50)

$\therefore$ the *denominator* $a+b=0$,
which is the required condition.

[In words · **If two lines are perpendicular, coefficient of x^2 + coefficient of $y^2=0$.**]

Note. The Converse is also seen to be true by working the algebra backwards.

(*ii*) If the lines are coincident, θ, the angle between them, $=0°$
$\therefore\ \tan\theta=\tan 0°=0$

i.e., $\frac{2\sqrt{h^2-ab}}{a+b}=0$ [Art. 50]

$\therefore\ \sqrt{h^2-ab}=0$

or $h^2-ab=0$, which is the required condition.

Note. The Converse is also seen to be true by working the algebra backwards.

Cor. 2. *To find the conditions that the lines $ax^2+2hxy+by^2=0$ may be real and different, real and coincident, or imaginary.*

The lines are real and different, real and coincident, or imaginary according as θ, the angle between them, is real, 0, or imaginary.

---

*[*R. B. 1950*]

i. e., according as $\tan\theta$ is real, 0, or imaginary,

i. e., $\dfrac{2\sqrt{h^2-ab}}{a+b}$ is real, 0, or imaginary, [Art. 50]

i. e., according as h^2-ab is +ve, 0, or −ve.

Note. If h^2-ab is negative, the lines are imaginary but they pass through a real point, the origin. [Art. 49]

Example. Shew that the two straight lines $y^2-2xy\sec\theta+x^2=0$ make an angle θ with one another. [*C. U. 1955*]

The equation of the lines is $y^2-2xy\sec\theta+x^2=0$,

or $x^2-2xy\sec\theta+y^2=0$. [Form $ax^2+2hxy+by^2=0$]

Comparing this with $ax^2+2hxy+by^2=0$,

here $a=1,\ h=\frac{1}{2}(-2\sec\theta)=-\sec\theta,\ b=1.$

[a=coeff. of x^2, $h=\frac{1}{2}$ coeff. of xy, b=coeff. of y^2]

$\therefore$ if ϕ* is the angle between the lines, then

$$\tan\phi=\frac{2\sqrt{h^2-ab}}{a+b}\ \text{[Art. 50]}$$

$$=\frac{2\sqrt{\sec^2\theta-1.1}}{1+1}=\frac{2\tan\theta}{2}=\tan\theta$$

$\therefore\quad \phi=\theta.$

EXAMPLES

Find what straight lines are represented by the following equations, and determine the angle between them (Exs. 1—2) :—

1. (*i*) $\mathbf{x^2-7xy+12y^2=0.}$ [*B.H.U. 1951*]

(*ii*) $\mathbf{4x^2+24xy+11y^2=0.}$

2. (*i*) $\mathbf{x^2+2xy\sec\theta+y^2=0.}$ [*R.B. 1943*]

(*ii*) $x^2+2xy\cot 2\alpha-y^2=0.$

(*iii*) $x^2-2xy\cot\theta+y^2=0.$

3. Shew that each of the following equations represents a pair of straight lines, and find the angle between each pair :

(*i*) $\mathbf{x^2-5xy+4y^2=0.}$ [*P.U.B. 1935*] (*ii*) $x^2-9y^2=0.$

4. Shew that the two straight lines

$$x^2(\tan^2\theta+\cos^2\theta)+2xy\tan\theta+y^2\sin^2\theta=0$$

make with the x-axis angles such that the difference of their tangents is 2.

[**Hint.** Let the two st. lines make angles ϕ_1, ϕ_2 with the x-axis ; to prove that $\tan\phi_1-\tan\phi_2=2$, *i.e.*, $m_1-m_2=2$. From the equation of the lines, find m_1+m_2, and m_1m_2 (use Art. 49, Cor. 1), and hence find m_1-m_2. Work in $\sin\theta$ and $\cos\theta$.]

SECTION III

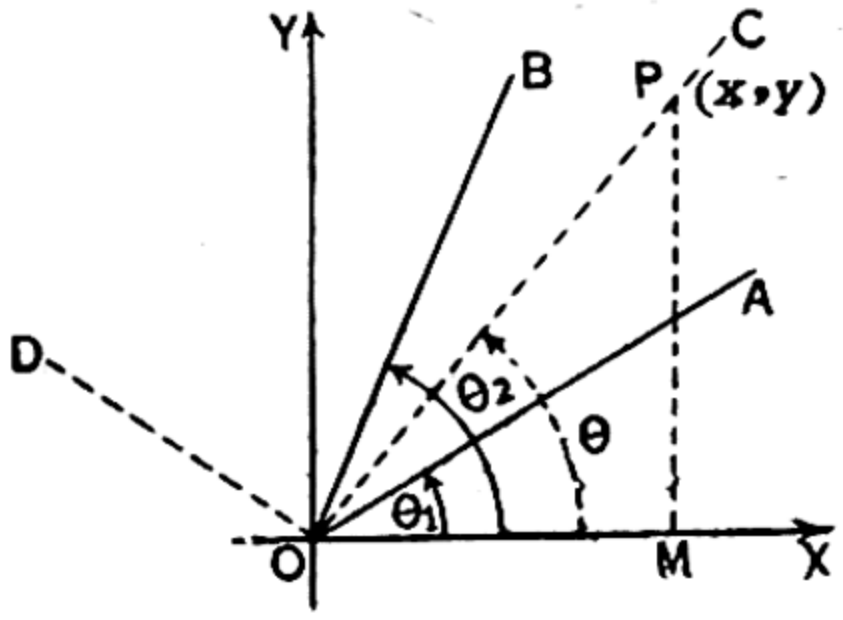
ANGLE-BISECTORS OF A PAIR OF STRAIGHT LINES THROUGH THE ORIGIN

51. Equation of the bisectors of the angle between two straight lines through the origin. To find the equation to the straight lines bisecting the angle between the straight lines given by $\mathbf{ax^2+2hxy+by^2=0}$†.

***Why to use ϕ (and not θ).** In order to avoid confusion with the given θ in the equation of the lines : $y^2-2xy\sec\theta+x^2=0$, we use ϕ (and *not* θ).

†[*J. & K.U.B. 1952*]

Method I.



Let OA, OB be the lines $ax^2+2hxy+by^2=0$, making $\angle$s θ_1, θ_2 with the x-axis,

so that $\tan\theta_1+\tan\theta_2=m_1+m_2=-\frac{2h}{b}$

and $\tan\theta_1 . \tan\theta_2=m_1m_2=\frac{a}{b}$ } [Art. 49, Cor. 1] ... (1)

Let OC, OD be the internal and external bisectors of $\angle$AOB.

Then $\because \angle AOC=\angle COB$

$\therefore \angle XOC-\angle XOA=\angle XOB-\angle XOC$

or $2\angle XOC=\angle XOA+\angle XOB$

$\therefore \angle XOC=\frac{1}{2}(\angle XOA+\angle XOB)=\frac{1}{2}(\theta_1+\theta_2)$,

and $\angle XOD=\angle XOC+\angle COD=\frac{1}{2}(\theta_1+\theta_2)+\frac{\pi}{2}$

[$\because$ OC, OD, being the internal and external bisectors of $\angle$AOB, are $\perp$]

Let θ be the angle which *either* bisector makes with the x-axis.

Then $\theta=\angle XOC=\frac{1}{2}(\theta_1+\theta_2)$,

or $\theta=\angle XOD=\frac{1}{2}(\theta_1+\theta_2)+\frac{\pi}{2}$ } (Proved above)

$\therefore 2\theta=\theta_1+\theta_2$ or $2\theta=\theta_1+\theta_2+\pi$

$\therefore \tan 2\theta=\tan(\theta_1+\theta_2)$,

or $\tan 2\theta=\tan(\theta_1+\theta_2+\pi)=\tan(\theta_1+\theta_2)$

$\therefore$ in *either* case, $\tan 2\theta=\tan(\theta_1+\theta_2)$

or $$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} \quad \ldots (2)$$

Let P (x, y) be *any* pt. on *either* bisector. From P draw PM $\perp$ on the x-axis. Then $\tan\theta=\frac{MP}{OM}=\frac{y}{x}$ ... (3)

Substituting from (3) and (1) in (2),

$$\frac{2\frac{y}{x}}{1-\frac{y^2}{x^2}}=\frac{-\frac{2h}{b}}{1-\frac{a}{b}}$$ [Cancel 2]

or $$\frac{xy}{x^2-y^2}=-\frac{h}{b-a}=\frac{h}{a-b}$$

$\therefore$ $$\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}}=\frac{\mathbf{xy}}{\mathbf{h}},$$

which is the required equation of the bisectors.

Method II. (Method of separate equations.)

The equation of the lines is $ax^2+2hxy+by^2=0$ ...(1)

Let the separate equations of the lines be

$$y=m_1x \quad ...(2), \text{ and } y=m_2x \quad ...(3)$$

Then $$m_1+m_2=-\frac{2h}{b}, \text{ and } m_1m_2=\frac{a}{b} \quad ...(4)$$ (Art. 49, Cor. 1)

The equations of the bisectors of the angles between the lines (2) and (3) are

$$\frac{y-m_1x}{\sqrt{1+m_1^2}}=\pm\frac{y-m_2x}{\sqrt{1+m_2^2}}$$ [Art. 46]

or $$\frac{y-m_1x}{\sqrt{1+m_1^2}}-\frac{y-m_2x}{\sqrt{1+m_2^2}}=0, \text{ and } \frac{y-m_1x}{\sqrt{1+m_1^2}}+\frac{y-m_2x}{\sqrt{1+m_2^2}}=0$$

$\therefore$ their combined equation is

$$\left(\frac{y-m_1x}{\sqrt{1+m_1^2}}-\frac{y-m_2x}{\sqrt{1+m_2^2}}\right)\left(\frac{y-m_1x}{\sqrt{1+m_1^2}}+\frac{y-m_2x}{\sqrt{1+m_2^2}}\right)=0$$ [Art. 47, Cor.]

or $$\frac{(y-m_1x)^2}{1+m_1^2}-\frac{(y-m_2x)^2}{1+m_2^2}=0$$

or $$(y^2-2m_1xy+m_1^2x^2)(1+m_2^2)-(y^2-2m_2xy+m_2^2x^2)(1+m_1^2)=0$$

or $$y^2[1+m_2^2-(1+m_1^2)]-2xy[m_1(1+m_2^2)-m_2(1+m_1^2)]+x^2[m_1^2(1+m_2^2)-m_2^2(1+m_1^2)]=0$$

or $$y^2(m_2^2-m_1^2)-2xy[m_1+m_1m_2^2-m_2-m_2m_1^2]+x^2[m_1^2-m_2^2]=0.$$

Factorising,

$$y^2(m_2+m_1)(m_2-m_1)-2xy[m_1-m_2+m_1m_2(m_2-m_1)]+x^2(m_1+m_2)(m_1-m_2)=0.$$

Dividing thro'out by m_2-m_1,

$$y^2(m_2+m_1)-2xy[-1+m_1m_2]+x^2(m_1+m_2)(-1)=0$$

or $$(y^2-x^2)(m_1+m_2)-2xy(m_1m_2-1)=0$$

[Substitute from (4)]

or $$(y^2-x^2)\left(-\frac{2h}{b}\right)-2xy\left(\frac{a}{b}-1\right)=0$$ [Cancel 2]

or $$(x^2-y^2)\frac{h}{b}-xy\left(\frac{a-b}{b}\right)=0$$

or $$(x^2-y^2)\,h=xy(a-b)$$

or $$\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}}=\frac{\mathbf{xy}}{\mathbf{h}},$$

which is the required equation of the bisectors.

[**N.B. The equation of the bisectors of the angles *between the lines* $ax^2+2hxy+by^2=0$, *is*** $\frac{\mathbf{x^2-y^2}}{\mathbf{a-b}}=\frac{\mathbf{xy}}{\mathbf{h}}$.]

EXAMPLES

1. *Find the equation of the straight lines bisecting the angles between the straight lines*

$$4x^2-16xy+7y^2=0.$$

The equation of the lines is $4x^2-16xy+7y^2=0$.
Comparing this with $ax^2+2hxy+by^2=0$,
here $a=4$, $h=\frac{1}{2}(-16)=-8$, $b=7$.
[a=coeff. of x^2, $h=\frac{1}{2}$ coeff. of xy, b=coeff. of y^2]
$\therefore$ the equation of the bisectors is

$$\frac{x^2-y^2}{4-7}=\frac{xy}{-8} \qquad \left[\frac{x^2-y^2}{a-b}=\frac{xy}{h} \quad \text{(Art. 51)}\right]$$

or $$\frac{x^2-y^2}{-3}=\frac{xy}{-8} \qquad [\text{ Cancel } (-1) \text{ }]$$

or $$8x^2-8y^2=3xy, \quad \text{or} \quad 8x^2-3xy-8y^2=0.$$

2. Find the equations of the straight lines bisecting the angles between the pairs of straight lines :

(*i*) $33x^2-13xy-6y^2=0$. (*ii*) $x^2-2xy\sec\theta+y^2=0$.

(*iii*) $x^2+2xy\cot\theta+y^2=0$.

3. If pairs of straight lines $x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisects the angles between the other pair, prove that $pq=-1$. [*J. & K.U.B. 1956*]

4. Prove that the pair of lines

$$a^2x^2+2h(a+b)xy+b^2y^2=0$$

is equally inclined to the pair

$$ax^2+2hxy+by^2=0.$$ [*P.U.B. 1939*]

The equation of the first pair of lines is

$$a^2x^2+2h(a+b)xy+b^2y^2=0 \quad \ldots(1)$$

and that of the second pair is

$$ax^2+2hxy+by^2=0 \quad \ldots(2)$$

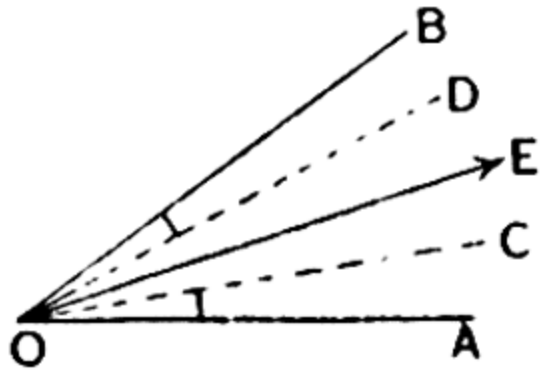
The equation of the bisectors of the angles between the pair (1) is

$$\frac{x^2-y^2}{a^2-b^2}=\frac{xy}{h(a+b)}, \text{ or } \frac{x^2-y^2}{(a+b)(a-b)}=\frac{xy}{h(a+b)} \qquad [\text{ Cancel } a+b \text{ }]$$

or $$\frac{x^2-y^2}{a-b}=\frac{xy}{h},$$

which is *the same as the equation of the bisectors of the angles between the pair* (*2*).

Let OA, OB be the lines of the pair (1), and OC, OD those of the pair (2). Let OE be the common bisector of $\angle$s AOB and COD.



Then $\angle AOE=\angle EOB$ ...(3)
and $\angle COE=\angle EOD$ ...(4)
Subtracting (4) from (3),

$$\angle AOC=\angle DOB,$$

i.e., OC, OD are equally inclined to OA, OB.

5. Show that the angle between one of the lines given by

$$ax^2+2hxy+by^2=0,$$

and one of the lines

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)=0,$$

is equal to the angle between the other two lines of the system.

[*P.U.B. 1944S*]

SECTION IV

GENERAL EQUATION OF THE SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES

52. Condition for the general equation of the second degree to represent two straight lines. To find the condition that the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0^*$$

may represent two straight lines.†

The equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots(1)$$

Writing it as a quadratic in x,

$$ax^2+2x(hy+g)+(by^2+2fy+c)=0.$$

Solving, $$x=\frac{-2(hy+g)\pm\sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$

$$=\frac{-(hy+g)\pm\sqrt{(h^2-ab)y^2+2(gh-af)y+(g^2-ac)}}{a} \qquad \ldots(2)$$

If (1) represents two st. lines, (2) gives two equations of the first degree in x and y.

$\therefore$ the expression under the square root in (2), *viz.*,

$(h^2-ab)y^2+2(gh-af)y+(g^2-ac)$ is a perfect square

$\therefore$ $\quad 4(gh-af)^2-4(h^2-ab)(g^2-ac)=0$ ["b^2-4ac"$=0$]

[Cancel 4]

or $\quad g^2h^2-2afgh+a^2f^2-(g^2h^2-ach^2-abg^2+a^2bc)=0$

or $\quad g^2h^2-2afgh+a^2f^2-g^2h^2+ach^2+abg^2-a^2bc=0$

[Cancel g^2h^2]

or $\quad -a(abc+2fgh-af^2-bg^2-ch^2)=0.$

[$\because$ $a\neq0$]

Dividing by $-a$,

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0,}$$

which is the required condition.

---

*Called the *general equation of the second degree* because it contains *all* possible terms in x and y of the *second* and lower degrees.

***Why to take these coefficients.** The general *homogeneous* equation of the second degree in x, y, z in Analytical Solid Geometry is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \ldots(1)$$

in which the coefficients occur in the alphabetical order, the letters d and e being omitted because d is reserved for differentiation and e for the base of Naperian logarithms. Putting $z=1$ in (1), we get

$ax^2+by^2+c+2fy+2gx+2hxy=0$, or $ax^2+2hxy+by^2+2gx+2fy+c=0$.

†[*J. & K. U. B. 1955*]

Note 1. Discriminant. Def. The expression **$abc+2fgh-af^2-bg^2-ch^2$** is called the **discriminant** of the general equation of the second degree.

Note 2. In obtaining the above condition (Art. 52) it has been assumed that $a \neq 0$. But the condition still holds (*i*) if $a=0$ but $b \neq 0$, or (*ii*) if $a=0$, $b=0$, but $h \neq 0$.

****Proof of case** (*i*). If $a=0$, but $b \neq 0$, the equation (1) becomes

$$2hxy+by^2+2gx+2fy+c=0 \quad \ldots(3)$$

Writing it as a quadratic in y,

$$by^2+2y(hx+f)+(2gx+c)=0.$$

Solving,

$$y=\frac{-2(hx+f)\pm\sqrt{4(hx+f)^2-4b(2gx+c)}}{2b}$$

$$=\frac{-(hx+f)\pm\sqrt{h^2x^2+2(hf-bg)x+(f^2-bc)}}{b} \quad \ldots(4)$$

If (3) represents two st. lines, then (4) gives two equations of the first degree in x and y.

$\therefore$ the expression under the square root in (4), *viz.*,
$h^2x^2+2(hf-bg)x+(f^2-bc)$ is a perfect square

$\therefore$ $4(hf-bg)^2-4h^2(f^2-bc)=0$ ["$b^2-4ac=0$"]

[Cancel 4]

or $h^2f^2-2bfgh+b^2g^2-h^2f^2+bch^2=0$

[Cancel h^2f^2]

or $-b(2fgh-bg^2-ch^2)=0.$

[$\because$ $b \neq 0$]

Dividing by $-b$,

$$2fgh-bg^2-ch^2=0$$

or $abc+2fgh-af^2-bg^2-ch^2=0.$ [$\because$ $a=0$]

****Proof of case** (*ii*). If $a=0$, $b=0$, but $h \neq 0$, the equation (1) becomes

$$2hxy+2gx+2fy+c=0 \quad \ldots(5)$$

Factorising, $2x(hy+g)+2\frac{f}{h}(hy+g)-\frac{2fg}{h}+c=0$

or, transposing, $2(hy+g)\left(x+\frac{f}{h}\right)=\frac{2fg}{h}-c \quad \ldots(6)$

If (5) represents two st. lines, then (6) gives two equations of the first degree in x and y, $\therefore$ R.H.S. of (6)$=0$

i.e., $\frac{2fg}{h}-c=0$, or $2fg-ch=0$

or, multiplying by h, $2fgh-ch^2=0$

or $abc+2fgh-af^2-bg^2-ch^2=0.$ [$\because$ $a=0$, $b=0$]

---

*__How to write this step.__ In order to take the common factor $hy+g$ as from the first two terms [$2hxy+2gx=2x(hy+g)$], we write

$2fy=2\frac{f}{h}(hy+g)-\frac{2f}{h}g.$ [Adding and subtracting $\frac{2fg}{h}$]

Note 3. If $a=0$, $b=0$, $h=0$, the equation (1) becomes $2gx+2fy+c=0$, which is *not* of the second degree and is, therefore, out of question.

Cor. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, the equation of the lines through the origin parallel to them is $ax^2+2hxy+by^2=0$ (i.e., second degree terms=0).

The equation of the lines is

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots (1)$$

Let the separate equations of the lines be

$lx+my+n=0$ ...(2), and $l'x+m'y+n'=0$ ... (3)

Then $ax^2+2hxy+by^2+2gx+2fy+c=(lx+my+n)(l'x+m'y+n')$ (identically).

Equating coeffs. of like terms on both sides,

$$\left.\begin{array}{ll} [\text{ Equating coeffs. of } x^2\ ] & ll'=a, \\ [\quad\ ,, \quad\ ,, \quad\ ,, \ xy\ ] & lm'+l'm=2h, \\ [\quad\ ,, \quad\ ,, \quad\ ,, \ y^2\ ] & mm'=b, \end{array}\right\} \ldots (4)$$

$$\begin{array}{ll} [\quad\ ,, \quad\ ,, \quad\ ,, \ x\ ] & nl'+n'l=2g, \\ [\quad\ ,, \quad\ ,, \quad\ ,, \ y\ ] & mn'+m'n=2f, \\ [\quad\ ,, \text{ constant terms}\ ] & nn'=c. \end{array}$$

The equations of the lines thro' the origin ‖ to the lines (2) and (3) are $lx+my=0$, and $l'x+m'y=0$.

$\therefore$ their combined equation is $(lx+my)(l'x+m'y)=0$ [Art. 47, Cor.]

or $ll'x^2+(lm'+l'm)xy+mm'y^2=0$ [Substitute from (4)]

or $\mathbf{ax^2+2hxy+by^2=0}$ **(i.e., second degree terms=0).**

Example 1. Prove that the equation

$$\mathbf{6x^2+5xy-4y^2+7x+13y-3=0}$$

represents two straight lines, and find the angle between them. [*P. U. B.*]

Method I. [Method of solving as quadratic.]

(*a*) The equation is

$$6x^2+5xy-4y^2+7x+13y-3=0 \quad \ldots (1)$$

Writing it as a quadratic in x, [*Note this step*]

$$6x^2+x(5y+7)-(4y^2-13y+3)=0.$$

Solving,

$$x=\frac{-(5y+7)\pm\sqrt{(5y+7)^2+24(4y^2-13y+3)}}{12}$$

$$=\frac{-(5y+7)\pm\sqrt{25y^2+70y+49+96y^2-312y+72}}{12}$$

$$=\frac{-(5y+7)\pm\sqrt{121y^2-242y+121}}{12}$$

$$=\frac{-(5y+7)\pm11\sqrt{y^2-2y+1}}{12}=\frac{-(5y+7)\pm11\sqrt{(y-1)^2}}{12}$$

$$=\frac{-(5y+7)\pm11(y-1)}{12}=\frac{6y-18}{12}, \text{ or } \frac{-16y+4}{12}$$

$$=\frac{y-3}{2}, \text{ or } \frac{-4y+1}{3}.$$

$\therefore$ (1) represents the two st. lines
$2x-y+3=0$ ... (2), and $3x+4y-1=0$ ... (3)

(*b*) Here $m_1 = -\frac{2}{-1}$ $\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y} \text{ (Art. 25, Cor.)}\right]$

$=2,$

$m_2 = -\frac{3}{4},$

$\therefore$ if θ is the angle between the lines (2) and (3), then

$$\tan\theta = \frac{2-(-\frac{3}{4})}{1+2(-\frac{3}{4})} \qquad \left[\frac{m_1-m_2}{1+m_1m_2}\right]$$

$$= \frac{2+\frac{3}{4}}{1-\frac{3}{2}} = \frac{\frac{11}{4}}{-\frac{1}{2}} = -\frac{11}{4}\times 2 = -\frac{11}{2} \quad \ldots(4)$$

$\therefore$ if ϕ is the acute angle between the lines, then changing the sign in (4) [Art. 33-I, Note 2, end],

$$\tan\phi = \frac{11}{2}, \quad \therefore \quad \phi = \tan^{-1}\frac{11}{2}.$$

Method II. [Method of condition for two straight lines.]

(*a*) The equation is

$$6x^2+5xy-4y^2+7x+13y-3=0 \quad \ldots (1)$$

Comparing this with

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

here $a=6,\ h=\frac{1}{2}(5),\ b=-4,\ g=\frac{1}{2}(7),\ f=\frac{1}{2}(13),\ c=-3,$

[a=coeff. of x^2, $h=\frac{1}{2}$ coeff. of xy, b=coeff. of y^2,
$g=\frac{1}{2}$ coeff. of x, $f=\frac{1}{2}$ coeff. of y, c=constant term]

$\therefore$ $abc+2fgh-af^2-bg^2-ch^2$

$$=6(-4)(-3)+2\left(\frac{13}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)-6\left(\frac{13}{2}\right)^2-(-4)\left(\frac{7}{2}\right)^2-(-3)\left(\frac{5}{2}\right)^2$$

$$=72+\frac{455}{4}-\frac{507}{2}+49+\frac{75}{4}=121+\frac{455-1014+75}{4}$$

$$=121-\frac{484}{4}=0.$$

1014
530
484

$\therefore$ (1) represents two st. lines. [Art. 52]

(*b*) *The angle between the lines (1)*
= the angle between the lines thro' the origin ∥ to them.

Now the equation of the lines thro' the origin ∥ to the lines (1) is
$6x^2+5xy-4y^2=0$ ...(2) [Second degree terms=0 (Art. 52, Cor.)]
Comparing this with $ax^2+2hxy+by^2=0$,
here $a=6,\ h=\frac{1}{2}(5),\ b=-4$

$\therefore$ if θ is the angle between the lines,

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{\frac{25}{4}-6(-4)}}{6-4} = \frac{2(\frac{11}{2})}{2} = \frac{11}{2}$$

$\therefore$ $\theta = \tan^{-1}\frac{11}{2}$.

[**Comparison of the two methods.** Method I (Writing the equation as a quadratic in x (or y) and solving it) enables us not only to prove that the given equation represents two straight lines, but also to find *their separate equations* from which we can find other things regarding the lines, *e.g.*, the angle between them or their point of intersection etc. (See Ex. I following.)]

Example 2. Find the value of h so that the equation

$$12x^2+2hxy+2y^2+11x-5y+2=0$$

may represent two straight lines. [*C. U.*]

The equation is

$$12x^2+2hxy+2y^2+11x-5y+2=0.$$

Comparing this with

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

here $a=12$, "h"$=\frac{1}{2}(2h)=h$, $b=2$, $g=\frac{1}{2}(11)$, $f=\frac{1}{2}(-5)$, $c=2$.

[a=coeff. of x^2, $h=\frac{1}{2}$ coeff. of xy, b=coeff. of y^2, $g=\frac{1}{2}$ coeff. of x, $f=\frac{1}{2}$ coeff. of y, c=constant term]

If (1) represents two st. lines, then

$$abc+2fgh-af^2-bg^2-ch^2=0. \qquad [\text{Art. 52}]$$

$$\therefore\ 12(2)(2)+2\left(-\frac{5}{2}\right)\left(\frac{11}{2}\right)h-12\left(-\frac{5}{2}\right)^2-2\left(\frac{11}{2}\right)^2-2h^2=0$$

or $48-\frac{55}{2}h-75-\frac{121}{2}-2h^2=0$ $\quad\left|-27-\frac{121}{2}\right.$

$\left.=\frac{-54-121}{2}=-\frac{175}{2}\right.$

or $-\frac{175}{2}-\frac{55}{2}h-2h^2=0$

or $4h^2+55h+175=0$

$$\therefore\ h=\frac{-55\pm\sqrt{3025-2800}}{8}=\frac{-55\pm\sqrt{225}}{8}$$

$$=\frac{-55\pm15}{8}=-\frac{40}{8},\ \text{or}\ -\frac{70}{8}$$

$$\therefore\ h=-5,\ \text{or}\ -\frac{35}{4}.$$

EXAMPLES

Shew that the following equations represent two straight lines; find also their point of intersection and the angle between them (Exs. 1—2) :

1. (*i*) $x^2-5xy+4y^2+4x-y-5=0$. [*A. U.*]

(*ii*) $6y^2-xy-x^2+10y+4=0$.

2. (*i*) $3y^2-8xy-3x^2-29x+3y-18=0$. [*P.U.B. 1938*]

(*ii*) $xy-2x-3y+6=0$.

[**Hint.** (*ii*) Factorise.]

3. Prove that the equation $x^2+6xy+9y^2+4x+12y-5=0$ represents two parallel lines, and find the distance between them. [*R.B. 1950*]

4. Find the value of k so that the following equations may represent pairs of straight lines :

(*i*) $x^2+kxy+y^2-5x+7y+6=0$.

(*ii*) $kxy-4x+3y-12=0$.

(*iii*) $2x^2-xy-y^2+kx-6y-9=0$.

5. Find the relations between the coefficients of the equations

(*i*) $ax^2+by^2+cx+cy=0$,

and (*ii*) $ay^2+bxy+cy+dx=0$,

so that each of them may represent a pair of straight lines.

6. (*a*) **Find the angle between two straight lines represented by the general equation of the second degree.** [*J. & K. U. 1956*]

(*b*) **For what value of λ does the equation**

$$12x^2-10xy+2y^2+11x-5y+\lambda=0$$

represent two straight lines ? Show that if the equation represents straight lines, the angle between them is $\tan^{-1}\frac{1}{7}$. [*P. U. B. 1948 Em.*]

7. For what value of λ does the equation

$$3x^2+9xy+\lambda y^2+3x+3y+3=0$$

represent two straight lines ?

Are these lines real or imaginary ?

[**Hint.** It will be found that $\lambda=7$. The lines are real or imaginary according as the lines thro' the origin $\parallel$ to them are real or imaginary. Now the equation of the lines thro' the origin $\parallel$ to the given lines is

$3x^2+9xy+7y^2=0$. [Second degree terms $=0$ (Art. 52, Cor.)]

Use Art. 50, Cor. 2.]

8. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent two straight lines, prove that the square of the distance of their point of intersection from the origin is $\frac{c(a+b)-f^2-g^2}{ab-h^2}$. [*P(P). U. B. 1948*]

[**Hint.** See Art. 52, Cor. It will be found that the pt. of intersection of the lines (2) and (3) is $\left(\frac{mn'-m'n}{lm'-l'm}, \frac{nl'-n'l}{lm'-l'm}\right)$.

$\therefore$ square of the distance of this pt. from the origin (0, 0)

$$=\frac{(mn'-m'n)^2+(nl'-n'l)^2}{(lm'-l'm)^2} \quad \ldots(5)$$

Now $(mn'-m'n)^2=(mn'+m'n)^2-4mm'nn'$

[Substitute from (4) etc.]

$=(2f)^2-4bc$, and so on.

Substitute these values in (5).]

****9. Prove that the general equation**

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two parallel straight lines if

$$h^2=ab \text{ and } bg^2=af^2.$$

Prove also that the distance between them is

$$2\sqrt{\frac{g^2-ac}{a(a+b)}}.$$ [*A. B. 1951*]

[**Hint.** The equation of the lines is

$ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(1)

Let the separate equations of the $\parallel$ lines be

$lx+my+n=0$ ...(2), and $lx+my+n'=0$ ...(3)

Then $ax^2+2hxy+by^2+2gx+2fy+c$

$=(lx+my+n)(lx+my+n')$ (identically).

Equating coeffs. of like terms on both sides (proceed as in Art. 52, Cor.), $l^2=a$, $2lm=2h$, $m^2=b$, $nl+n'l=2g$, $mn'+mn=2f$, $nn'=c$.

$\therefore\ h^2=l^2m^2$, also $ab=l^2m^2$, $\therefore\ h^2=ab$;

$bg^2=\dfrac{m^2l^2(n+n')^2}{4}$, also $af^2=\dfrac{l^2m^2(n+n')^2}{4}$, $\therefore\ bg^2=af^2$.

It will be found that the distance between the ‖ lines (2) and (3)

$$=\frac{n'-n}{\sqrt{l^2+m^2}}=\frac{\sqrt{(n+n')^2-4nn'}}{\sqrt{l^2+m^2}} \quad \ldots(4)$$

Now $n+n'=\dfrac{2g}{l}=\dfrac{2g}{\sqrt{a}}$, $nn'=c$, $l^2=a$, $m^2=b$.

Substitute in (4).]

53. Equation of the lines joining the origin to the points of intersection of a straight line and a curve. *To find the equation of the two straight lines joining the origin to the points of intersection of the straight line* $lx+my=n$
and the curve $ax^2+2hxy+by^2+2gx+2fy+c=0$.

The equation of the line is $lx+my=n$

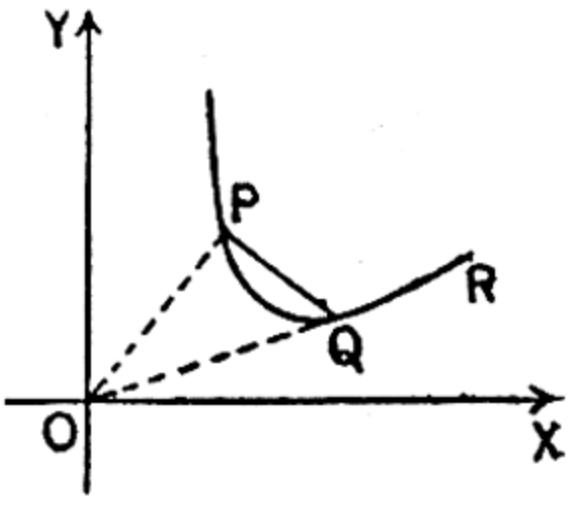
or, dividing by n, $\dfrac{lx+my}{n}=1$ ...(1) [R.H.S.=1 (*Note this step*)]

The equation of the curve is

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots(2)$$

[Let PQ be the line and PQR the curve, and P, Q the pts. of intersection of the line and the curve.

To find the equation of OP, OQ.]



Making (2) homogeneous by means of (1),

$$ax^2+2hxy+by^2+2gx\left(\frac{lx+my}{n}\right)+2fy\left(\frac{lx+my}{n}\right)+c\left(\frac{lx+my}{n}\right)^2=0 \quad \ldots(3)$$

(*i*) It is a homogeneous equation of the second degree,
$\therefore$ it represents two st. lines thro' the origin. [Art. 49]

(*ii*) The co-ordinates of the pts. which satisfy both (1) and (2), also satisfy (3)

[$\because$ substituting from (1) in (3), we get
$ax^2+2hxy+by^2+2gx+2fy+c=0$, which is true (From (2))]

$\therefore$ the pts. of intersection of the line (1) and the curve (2) lie on (3).

$\therefore$ (3) is the equation of the two st. lines joining the origin to the pts. of intersection of the line (1) and the curve (2).

[**Rule to find the equation of the two straight lines joining the origin to the points of intersection of a straight line and a curve (whose equation is of the second degree) :**

(*i*) *Write the equation of the line so that the constant term occurs on the R.H.S. and* $=1$. **(Note this step)**

(*ii*) *Make the equation of the curve homogeneous by means of the equation of the line written as in step* (*i*). *The resulting equation is the required equation.*]

Example. Find the lines joining the origin to the points of intersection of $3x^2+4xy-4x+1=0$ and $2x+y-1=0$, and shew that they are perpendicular. [*P. U. B.*]

The equation of the curve is

$$3x^2+4xy-4x+1=0 \quad \ldots(1)$$

and that of the line is $2x+y-1=0$

or $$2x+y=1 \quad \ldots(2)$$

[R.H.S.$=1$ (*Note this step*)]

Making (1) homogeneous by means of (2),

$$3x^2+4xy-4x(2x+y)+1(2x+y)^2=0$$

or $$3x^2+4xy-8x^2-4xy+4x^2+4xy+y^2=0$$

or $$-x^2+4xy+y^2=0, \text{ or } x^2-4xy-y^2=0,$$

which is the equation of the lines joining the origin to the pts. of intersection of the curve (1) and the line (2).

Comparing this with $ax^2+2hxy+by^2=0$,

here $a=1$, $h=\frac{1}{2}(-4)=-2$, $b=-1$

$\therefore\ a+b=1+(-1)=1-1=0.$

$\therefore$ the lines are $\perp$ (Art. 50, Cor. 1, (*i*)).

EXAMPLES

1. Shew that the lines joining the origin to the points common to $3x^2+5xy-3y^2+2x+3y=0$ and $3x-2y=1$ are at right angles. [*J. & K. U. B. 1954*]

2. Find the equation of the pair of lines joining the origin of co-ordinates to the points of intersection of the line $y=mx+c$ with the curve $x^2+y^2=a^2$. Prove that they are perpendicular if

$$2c^2=a^2(1+m^2).$$ [*A. U.*]

3. Prove that the angle between the straight lines joining the origin to the intersections of the straight line $y=3x+2$ with the curve $x^2+2xy+3y^2+4x+8y-11=0$ is $\tan^{-1}\frac{2\sqrt{2}}{3}$. [*C. U. 1945*]

4. Shew that the straight lines joining the origin to the points of intersection of the two curves

$$ax^2+2hxy+by^2+2gx=0,$$

and

$$a'x^2+2h'xy+b'y^2+2g'x=0,$$

will be at right angles to one another, if

$$g'(a+b)=g(a'+b').$$ [*A. U. 1945*]

[**Hint.** The equations of the curves are

$$ax^2+2hxy+by^2+2gx=0 \quad \ldots(1)$$

and $$a'x^2+2h'xy+b'y^2+2g'x=0 \quad \ldots(2)$$

To make (1) homogeneous by means of (2), eliminate the term in x from (1) and (2) (by multiplying (1) by g', (2) by g, and subtracting).]

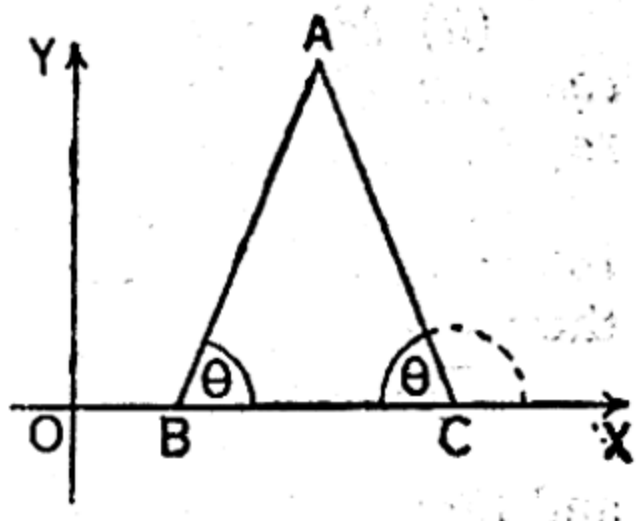
5. Condition for two straight lines to make equal angles with the x-axis. The condition that the two straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$* may make equal angles with the x-axis is

$m_1=-m_2$ (*i.e.*, the slope of one line=negative slope of the other).

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*Of course, not parallel.

Let AB, AC be the lines $y=m_1x+c_1$, $y=m_2x+c_2$ meeting the x-axis in B, C, and making an angle θ with it.



Then $m_1=\tan \text{XBA}=\tan \theta$,

$m_2=\tan \text{XCA}=\tan (\pi-\angle\text{ACB})$

$=\tan (\pi-\theta)=-\tan \theta$,

$\therefore \quad \mathbf{m_1=-m_2}$.

[**Aid to memory.** The condition, that the lines $y=m_1x+c_1$ and $y=m_2x+c_2$ may be parallel, is $\mathbf{m_1=m_2}$.

But the condition that they may make equal angles with the x-axis is $\mathbf{m_1=-m_2}$.]

6. Prove that the straight lines joining the origin to the points of intersection of the straight line $x-y-1=0$ and the curve

$$5x^2+12xy-6y^2+4x-2y+3=0$$

make equal angles with the axes. [*P. U. B. 1937*]

MISCELLANEOUS EXAMPLES ON CHAPTER IV

1. Find the area of the triangle bounded by the lines $y^2-9xy+18x^2=0$ and $y=9$. [*A. U. 1942*]

2. Prove that the equations to the straight lines passing through the origin which make an angle α with the straight line $x+y=0$ are given by the equation $x^2+2xy \sec 2\alpha+y^2=0$.

[**Hint.** Let the equation of a line thro' the origin making an angle α with $x+y=0$ (slope$=-1$) be $y=mx$ ...(1)

Then $\tan \alpha=\pm\dfrac{m-(-1)}{1+m(-1)}=\pm\dfrac{1+m}{1-m}$ ...(2)

Eliminating m from (1) and (2) [by substituting its value $\left(m=\dfrac{y}{x}\right)$ from (1) in (2)], $\tan \alpha=\pm\dfrac{x+y}{x-y}$, or $\dfrac{\sin \alpha}{\cos \alpha}=\pm\dfrac{x+y}{x-y}$.]

3. Prove that one of the lines $ax^2+2hxy+by^2=0$ will bisect an angle between the co-ordinate axes if $(a+b)^2=4h^2$. [*B. U. 1942*]

[**Hint.** The equation of the lines is $ax^2+2hxy+by^2=0$ ...(1)

It will be found that the equation of the bisectors of an angle between the axes is $x^2-y^2=0$ ...(2)

Find the condition that one of the lines (1) may coincide with one of the lines (2). (Proceed as in solved Ex., Art. 49.)]

4. Find the equation of the lines through the origin perpendicular to the lines $2x^2+3xy+y^2=0$. [*A. U. 1948*]

5. Prove that the product of the perpendiculars from (p, q) on the straight lines given by $\mathbf{ax^2+2hxy+by^2=0}$ is

$$\frac{\mathbf{ap^2+2hpq+bq^2}}{\sqrt{\mathbf{(a-b)^2+4h^2}}}.$$ [*P. U. B. 1951S*]

6. Show that the equation $\lambda y^2+(1-\lambda^2)xy-\lambda x^2=0$, where λ is a constant, represents a pair of perpendicular straight lines. [*A. U. 1938*]

7. Show that the pairs of lines $y^2-4xy-x^2=0$, $y^2+xy-x^2=0$ are such that each pair bisects the angles between the other pair. [*B.U. 1931*]

8. Shew that the pair of lines $2x^2+6xy+y^2=0$ is equally inclined to the lines $4x^2+18xy+y^2=0$. [*A. U. 1948*]

9. Show that the equation $x^2-5xy+4y^2+3x-4=0$ represents a pair of straight lines and find the angle between them. [*A. U. 1935*]

10. Find the lines represented by the equation

$$6x^2-5xy-6y^2+16x+2y+8=0$$

and determine the angle between them. [*D. U. 1939*]

11. Prove that $y^2-4xy+4x^2-6x+3y+2=0$ represents a pair of parallel straight lines, and find their equations. [*A. U. 1947*]

12. Show that $8x^2+8xy+2y^2+26x+13y+15=0$ represents a pair of parallel lines. Find also the perpendicular distance between the lines. [*A. U. 1948*]

13. For what value of k, does the equation

$$2x^2+3xy+y^2-3x+ky-2=0$$

represent a pair of straight lines ? [*B. U. 1923*]

14. For what value of k will the equation

$$kx^2-10xy+12y^2+5x-16y-3=0$$

represent a pair of straight lines ? Also find the equations of the lines. [*J. & K. U. 1949*]

15. If $6x^2-11xy-10y^2+19y+c=0$ represents two straight lines, find the equations of the lines and the tangent of the angle between them. [*R. B. 1948*]

Find also the point of intersection of the two lines. [*B. U.*]

16. Find the equation to the pair of straight lines through the origin perpendicular to the pair of straight lines given by

$$2x^2+5xy+2y^2+10x+5y=0.$$ [*C. U. 1954*]

17. Find the equation of the pair of lines joining the origin of co-ordinates to the points of intersection of the line $y=x+2$ with the curve $x^2+y^2=4$. [*A. U. 1947*]

18. Find the equation of the straight lines joining the origin to the points of intersection of the straight line $x+y+2=0$ with the curve $x^2+xy+y^2+x+3y+1=0$. [*A. U. 1936*]

19. Show that the straight lines joining the origin to the points of intersection of the line $x+y=1$ with the circle

$$4x^2+4y^2+4x-2y-5=0$$

are at right angles to each other. [*R. B. 1943*]

20. If the lines joining the origin to the points of intersection of $y=mx+1$ with $x^2+y^2=1$ are perpendicular, find the value of m. [*A. U. 1948*]

****21.** Prove that the straight lines joining the origin to the intersections of the straight line $kx+hy=2hk$ and the curve $(x-h)^2+(y-k)^2=a^2$ are perpendicular if $h^2+k^2=a^2$. [*C. U. 1953*]

CHAPTER V
THE CIRCLE
SECTION I
EQUATION OF A CIRCLE IN DIFFERENT FORMS

54. Circle. Def.* *A* **circle** *is the locus of a point which moves*† *so that its distance from a fixed point is constant.*

Defs. (*i*) The fixed point is called the **centre,**
and (*ii*) the constant distance is called the **radius** of the circle.

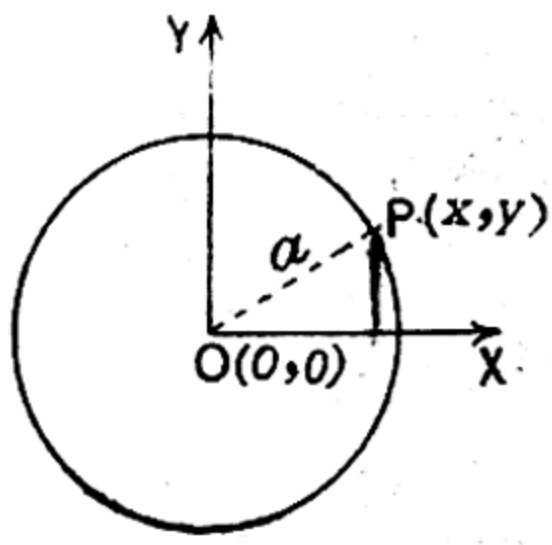
†**Note.** It is assumed that the point moves in a plane in which the fixed point lies.

A particular standard form
A circle whose centre is the origin.

55. Standard form. *To find the equation of a circle in the form* $x^2+y^2=a^2$.

Let O be the centre and a the radius of the circle.

Take the centre as origin and two ⊥ lines OX, OY thro' it as the axes.



(*i*) Let P(x, y) be *any* pt. on the circle. Join OP.

(*ii*) Then $OP=a$. [Def. (Art. 54)]

(*iii*) $\therefore \sqrt{(x-0)^2+(y-0)^2}=a$,

or $\sqrt{x^2+y^2}=a$

or, squaring, $\mathbf{x^2+y^2=a^2}$,

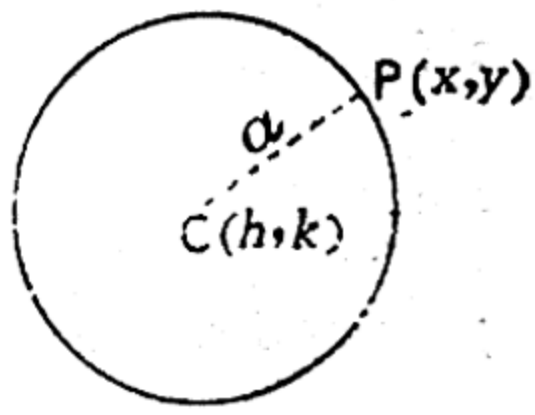
which is the required equation.

Note. Standard form. Since $x^2+y^2=a^2$ is the simplest form of the equation of a circle, it is called the **standard form.**

Three general standard forms
I. Central form.

56. Central form. To find the equation of a circle whose centre and radius are given.†

Let C(h, k) be the centre and a the radius of the circle.



(*i*) Let P(x, y) be *any* pt. on the circle. Join CP.

(*ii*) Then $CP=a$. [Def. (Art. 54)]

(*iii*) $\therefore \sqrt{(x-h)^2+(y-k)^2}=a$.

or, squaring, $\mathbf{(x-h)^2+(y-k)^2=a^2}$,

which is the required equation.

Note. Central form. Since in the equation $(x-h)^2+(y-k)^2=a^2$, (h, k) is the *centre* of the circle, this form of the equation of a circle is called the **central form.**

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*[*P. U. 1945*] †[*J. & K. U. 1952*]

Example. Find the equation of the circle whose centre is (−3, −4) and radius is 5.

The centre is (−3, −4) and radius=5,

∴ the equation of the circle is

$[x-(-3)]^2+[y-(-4)]^2=5^2$ $[(x-h)^2+(y-k)^2=a^2]$

or $(x+3)^2+(y+4)^2=5^2$ or $x^2+6x+9+y^2+8y+16=25$

or $x^2+y^2+6x+8y=0.$

EXAMPLES

1. (*a*) Find the equation of the circle whose centre is at the point (a, b) and whose radius is r. [*Bar. U. 1954*]

****(*b*) Obtain the equation of a circle in the form**

$$\mathbf{x^2+y^2+2gx+2fy+c=0.}$$ [*D. U. 1952*]

[**Sol.** (*b*) The equation of the circle whose centre is (*h*, *k*) and radius is *a*, is $(x-h)^2+(y-k)^2=a^2$

or $x^2-2hx+h^2+y^2-2ky+k^2=a^2$

or $x^2+y^2-2hx-2ky+(h^2+k^2-a^2)=0$

or $x^2+y^2+2gx+2fy+c=0,$

where $g=-h, f=-k, c=h^2+k^2-a^2.$

Note. The Converse of Ex. 1, (*b*) is *important* and is proved in Art. 57.]

2. Find the equation of a circle

(*i*) whose centre is (1, −2) and radius is 3.

(*ii*) whose centre is the point (a, b) and whose radius is $\sqrt{a^2+b^2}$. [*P. U.*]

3. Find the equation to the circle

(*i*) whose centre is (2, 3) and which passes through the point (5, 7). [*C. U. 1937*]

(*ii*) whose centre is the point (2, 3) and which passes through the intersection of the lines

$$\mathbf{3x-2y-1=0 \text{ and } 4x+y-27=0.}$$ [*C. U. 1947*]

II. General form.

57. General form. To prove that the equation

$$\mathbf{x^2+y^2+2gx+2fy+c=0}$$

represents a circle and to find its centre and radius.*

The equation is $x^2+y^2+2gx+2fy+c=0$ ...(1)

(*i*) Transposing, $(x^2+2gx)+(y^2+2fy)=-c.$

(Constant term on R. H. S.)

(*ii*) Completing the squares,†

$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c.$

[Adding g^2+f^2 to both sides]

or $(x+g)^2+(y+f)^2=g^2+f^2-c$

or $[x-(-g)]^2+[y-(-f)]^2=(\sqrt{g^2+f^2-c})^2.$

Comparing this with the equation of the circle

$(x-h)^2+(y-k)^2=a^2,$ [Art. 56]

here $h=-g, k=-f, a=\sqrt{g^2+f^2-c}.$

---

*[*N. U. 1953*]

†**Rule to "complete the square"** of an expression like x^2+2gx:

Add the square of half the coefficient of x, i.e., g^2.

$\therefore$ (1) represents a circle
whose centre is $(-\mathbf{g}, -\mathbf{f})$ [h, k]
and radius $=\sqrt{\mathbf{g^2+f^2-c}}$. [a]

Note 1. (*i*) If g^2+f^2-c is positive, the radius of the circle is real and $\therefore$ the circle is **real.**

(*ii*) If $g^2+f^2-c=0$, the radius of the circle is zero, and $\therefore$ the circle becomes a point coinciding with the point $(-g, -f)$; it is called a **point circle.**

(*iii*) If g^2+f^2-e is negative, the radius of the circle is imaginary, but its centre is real ; it is called an **imaginary* circle.**

[It is better *not* to say that "the circle does not exist" but to say that "the circle is imaginary," *i.e.*, the circle has a real centre and imaginary radius. The equation does not represent a geometrical locus, *i.e.*, the locus of the equation cannot be drawn in a Fig.]

Note 2. General form. Since the equation $x^2+y^2+2gx+2fy+c=0$ can, by a proper choice of g, f and c, be made to represent *any* circle, this form of the equation of a circle is called the **general form.**

Note 3. *A circle is determined by three conditions.* For, the general equation of a circle contains three constants (g, f and c) which are determined from the three equations obtained by using the three given conditions.

[Rule to write down the centre and radius of a circle whose equation is given :

(*i*) *Write the equation of the circle so that the coefficients of* x^2 *and* y^2 *are each* $=1$ *on the L.H.S. (R.H.S. being zero), by dividing the equation by the coefficient of* x^2 *or* y^2, *if necessary.* (*Note this step*)

(*ii*) *Compare this with* $x^2+y^2+2gx+2fy+c=0$, *and write down the values of* $g(=\frac{1}{2}$ *coefficient of* $x)$, $f(=\frac{1}{2}$ *coefficient of* $y)$,
$c(=$ *constant term*).

(*iii*) *Then the centre is* $(-g, -f)$,
and the radius $=\sqrt{g^2+f^2-c}$.]

Example. Find the co-ordinates of the centre and radius of the circle

(i) $\mathbf{x^2+y^2-10x-14y+58=0}$; [*P. U.*]
(ii) $\mathbf{4(x^2+y^2)+12ax-6ay-a^2=0}$. [*P. U.*]

(i) (*i*) The equation of the circle is $x^2+y^2-10x-14y+58=0$.
[Coeffs. of x^2 and y^2 already $=1$]

(*ii*) Comparing this with $x^2+y^2+2gx+2fy+c=0$,

here $g=\frac{1}{2}(-10)$ [$\frac{1}{2}$ coeff. of x]
$=-5$,
$f=\frac{1}{2}(-14)$ [$\frac{1}{2}$ coeff. of y]
$=-7$,
$c=58$. [Constant term]

(*iii*) $\therefore$ the centre is $[-(-5), -(-7)]$ [$-g, -f$]
i. e., $(5, 7)$,
and the radius $=\sqrt{(-5)^2+(-7)^2-58}$ [$\sqrt{g^2+f^2-c}$]
$=\sqrt{16}=4$.

---

*So called because it can *not* be drawn in a figure.

****Independent method.**

The equation of the circle is $x^2+y^2-10x-14y+58=0$.

[Coeffs. of x^2, y^2 already $=1$]

(*i*) Transposing, $(x^2-10x)+(y^2-14y)=-58$.

[Constant term on R.H.S.]

(*ii*) Completing the squares,

$(x^2-10x+25)+(y^2-14y+49)=25+49-58$

[Adding $25+49$ to both sides]

$=16$

or $(x-5)^2+(y-7)^2=(4)^2$

[Form $(x-h)^2+(y-k)^2=a^2$]

(*iii*) $\therefore$ the centre is $(5, 7)$, [h, k]

and the radius $=4$. [a]

(ii) The equation of the circle is

$4(x^2+y^2)+12ax-6ay-a^2=0$.

(*i*) *Dividing thro' out by 4,* $x^2+y^2+3ax-\frac{3a}{2}y-\frac{a^2}{4}=0$.

[Coeffs. of x^2 and $y^2=1$ (*Note this step*)]

(*ii*) Comparing this with $x^2+y^2+2gx+2fy+c=0$

here $g=\frac{1}{2}(3a)$ [$\frac{1}{2}$ coeff. of x]

$=\frac{3a}{2}$,

$f=\frac{1}{2}\left(-\frac{3a}{2}\right)$ [$\frac{1}{2}$ coeff. of y]

$=-\frac{3a}{4}$.

$c=-\frac{a^2}{4}$. [Constant term]

(*iii*) $\therefore$ the centre is $\left[-\frac{3a}{2}, -\left(-\frac{3a}{4}\right)\right]$ [$-g$, $-f$]

i.e., $\left(-\frac{3a}{2}, \frac{3a}{4}\right)$,

and the radius $=\sqrt{\left(\frac{3a}{2}\right)^2+\left(-\frac{3a}{4}\right)^2-\left(-\frac{a^2}{4}\right)}$ [$\sqrt{g^2+f^2-c}$]

$=\sqrt{\frac{9a^2}{4}+\frac{9a^2}{16}+\frac{a^2}{4}}=\sqrt{\frac{49a^2}{16}}=\frac{7a}{4}$.

EXAMPLES

1. (*a*) **Show that the equation $x^2+y^2+2gx+2fy+c=0$ represents a circle.** [*P. (P). U. 1955*]

(*b*) **Find the radius and the co-ordinates of the centre of the circle $x^2+y^2-8x-16y+78=0$.** [*D.U. 1931*]

2. (*a*) Obtain the radius and co-ordinates of the centre of the circle given by the equation $x^2+y^2+2gx+2fy+c=0$. [*D.U. 1953*]

(*b*) **Find the centres and the radii of the circles whose equations are**

(*i*) $7x^2+7y^2-4x-y=3$. [*P. U.*] (*ii*) $ax^2+ay^2=bx+cy$. [*P. U.*]

3. (*i*) **Show that the equation** $a(x^2+y^2)+2gx+2fy+c=0$ **represents a circle.**

(*ii*) **Find the radius and the co-ordinates of the centre.**

[*Pesh. U. 1955*]

[**Hint.** (*i*) Divide the equation by a, and proceed as in Art. 57.]

4. Tracing a circle. *Draw the circle* (*i*) $x^2+y^2=2x$. [*B.H.U.*]

(*ii*) $2x^2+2y^2=3y$. (*iii*) $x^2+y^2-5x+7y=0$. [*P. U.*]

[**Hint.** Find the centre and the radius.

Short-cut. Find the centre and plot it. Notice that the circle passes thro' the *origin* (put $x=0$, $y=0$ in the equation)

$\therefore$ *radius = join of the centre to the origin.*]

58. Conditions for a circle. *To find the conditions that an equation in x and y may represent a circle.*

The equation $x^2+y^2+2gx+2fy+c=0$ represents a circle. [Art. 57]

Multiplying the equation by A,

$Ax^2+Ay^2+2gAx+2fAy+cA=0$, also represents a circle.

$\therefore$ **the conditions for an equation to represent a circle are :**

(*i*) *The equation should be of the second degree in x and y.*

(*ii*) *The coefficients of* x^2 *and* y^2 *should be equal.*

(*iii*) *There should be no term involving the product xy.*

EXAMPLES

1. *A point moves so that the sum of the squares of its distances from two fixed points is constant* ($=2c^2$). *Prove that its locus is a circle.*

[*J. & K. U. 1953*]

2. Circle of Apollonius*. Show that the locus of a point such that the ratio of its distances from two given points is constant ($=k$) **is a circle.** [*P. U. 1930*]

59. Circle through three given points.

Example. Find the equation of the circle which passes through the points (1, 1), (2, −1) and (3, −2). [*P. U. 1941*]

(*i*) Let the required equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$

(*ii*) $\because$ it passes thro' (1, 1), these co-ordinates satisfy (1)

$\therefore$ $1+1+2g+2f+c=0$ or $2+2g+2f+c=0$ ...(2)

Again $\because$ it passes thro' (2, −1),

$\therefore$ $4+1+4g-2f+c=0$ or $5+4g-2f+c=0$ ...(3)

Also $\because$ it passes thro' (3, −2),

$\therefore$ $9+4+6g-4f+c=0$ or $13+6g-4f+c=0$ ...(4)

†Subtracting (3) from (2),

$-3-2g+4f=0$ or $2g-4f+3=0$ ...(5)

Subtracting (4) from (3),

$-8-2g+2f=0$ or $g-f+4=0$ ...(6)

Solving (5) and (6) by cross-multiplication,

See p. 14, footnote.

†**Why this step.** To eliminate c and get an equation in g and f only.

$$\begin{array}{ccccc} & \overset{g}{\frown} & & \overset{f}{\frown} & \overset{1}{\frown} \\ -4 & & 3 & & 2 & & -4 \\ -1 & \times & 4 & \times & 1 & \times & -1 \end{array}$$

$$\therefore \frac{g}{-16+3}=\frac{f}{3-8}=\frac{1}{-2+4} \quad \text{or} \quad \frac{g}{-13}=\frac{f}{-5}=\frac{1}{2}$$

$$\therefore g=-\frac{13}{2}, f=-\frac{5}{2}.$$

Substituting these values of g and f in (2),

$$2-13-5+c=0, \qquad \therefore \ c=16.$$

(*iii*) Substituting these values of g, f and c in (1),

$x^2+y^2-13x-5y+16=0$, which is the required equation.

[**Check.** This equation is satisfied by the co-ordinates of the three given pts. (1, 1), (2, −1), (3, −2), thus

$1+1-13-5+16=0$ or $0=0$, $4+1-26+5+16=0$ or $0=0$,

$9+4-39+10+16=0$ or $0=0$.]

[Rule to find the equation of a circle passing through three given points :

(*i*) *Let the required equation of the circle be*

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$

(*ii*) ∵ *it passes thro' the three given points,* ∴ *substitute their co-ordinates in the above equation in succession, and form three equations in g, f and c, and solve them.*

(*iii*) *Substitute these values of g, f and c in the equation of step* (*i*). *The result is the required equation.*

Note. The *Rule to find the equation of a circle determined by some other conditions* is exactly similar to the above, the only difference being that here in step (*ii*), (*instead of substituting the co-ordinates of the three points*) *we make the circle satisfy the three given conditions.*]

EXAMPLES

1. Find the equation of the circle which passes through the points :

(*i*) **(1, 2), (3, −4) and (5, −6) ;** [*J. & K. U.*]

(*ii*) **(a, 0), (−a, 0) and (0, b) ;**

(*iii*) **(0, 0), (a, 0) and (0, b) ; find its centre and radius.**

[*P.U. 1945S*]

2. Find the equation of a circle which passes through the points (−3, 4), (−2, 0), (1, 5). Also find the co-ordinates of the centre and the length of the radius of the circle obtained in the first part of this question. [*P.U. 1937*]

3. Circumcircle. Prove that the equation of the circle circumscribing the triangle formed by the lines $x+y=6$, $2x+y=4$ and $x+2y=5$ is $x^2+y^2-17x-19y+50=0$. [*J. & K. U. 1950*]

4. Find the equation of a circle which passes through the points (4, 1) and (6, 5) and has its centre on the line $4x+y=16$. [*P.U. 1931*]

[**Hint.** $(-g, -f)$ lies on $4x+y=16$, ∴ $-4g-f=16$.]

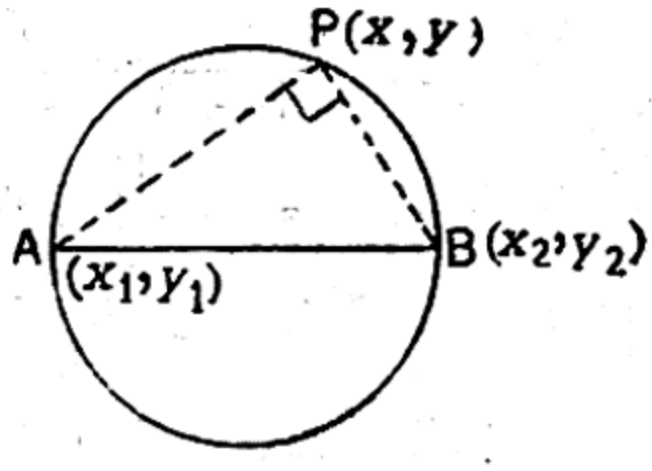
5. Find the equation of a circle which passes through the points (0, a), (h, k), and has its centre on the axis of x.

III. Diameter form.

60. Diameter form. To find the equation of the circle on the join of (x_1, y_1) and (x_2, y_2) as diameter.*

Let A, B be the pts. (x_1, y_1), (x_2, y_2).

(*i*) Let P (x, y) be *any* pt. on the circle. Join AP and BP.



(*ii*) Then $\angle APB$, *in a semicircle*, $=1$ *rt.* $\angle$, *i.e.*, AP $\perp$ BP ... (1)

(*iii*) Now the slope of AP $= \dfrac{y-y_1}{x-x_1}$ $\left[\dfrac{y_2-y_1}{x_2-x_1}\right]$

and the slope of BP $= \dfrac{y-y_2}{x-x_2}$.

$\therefore$ from (1), $\dfrac{y-y_1}{x-x_1} \cdot \dfrac{y-y_2}{x-x_2} = -1$ $[\, m_1 m_2 = -1 \,]$

or $(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$

or $\mathbf{(x-x_1)\,(x-x_2)+(y-y_1)\,(y-y_2)=0,}$

which is the required equation.

Note. Diameter form. Since in the equation $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$, (x_1, y_1) and (x_2, y_2) are the extremities of a *diameter* of the circle, this form of the equation of a circle may be called the **diameter form.**

EXAMPLES

1. (*a*) Obtain the equation of a circle when the co-ordinates of the extremities of a diameter are given. [*D.U. 1951*]

(*b*) The co-ordinates of the extremities of a line are (3, 4), (2, −7). Find the equation of the circle which has this line as its diameter. [*P(P).U. 1955*]

[**Sol.**† (*b*) The equation of the circle is (1) (3, 4) (2) (2, −7) $[(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$]

$(x-3)(x-2)+(y-4)\,[y-(-7)]=0$

or $(x-3)(x-2)+(y-4)(y+7)=0$

or $x^2-5x+6+y^2+3y-28=0$ or $x^2+y^2-5x+3y-22=0.$]

2. (*a*) Find the equation of the circle drawn on the line joining the points (a, b) and (c, d) as diameter. [*D.U. 1957*]

(*b*) Find the equation of the circle having for diameter the line joining the points (0, 1) and (1, 1). [*D.U.*]

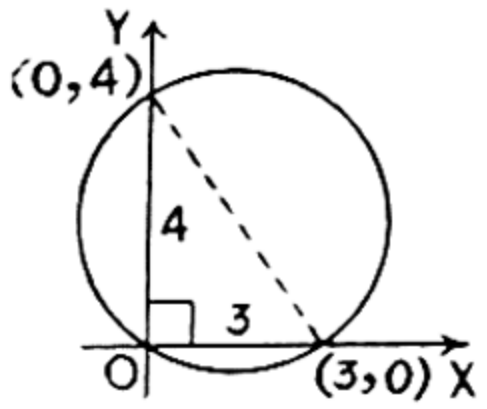
Find the radius and co-ordinates of the centre of this circle. [*D.U.*]

3. Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the axes. [*R. 1943*]

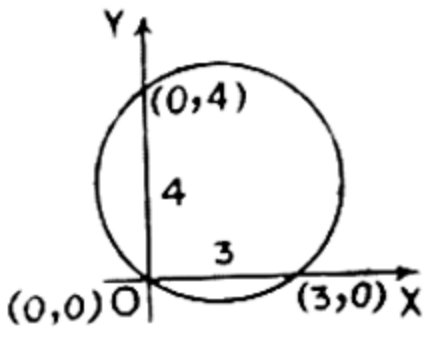
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* [*J. & K. U. 1954*]

† **Or thus : Hint.** The centre of the circle is the mid-pt. of the join of (3, 4) and (2, −7), and radius=half the distance between (3, 4) and (2, −7).



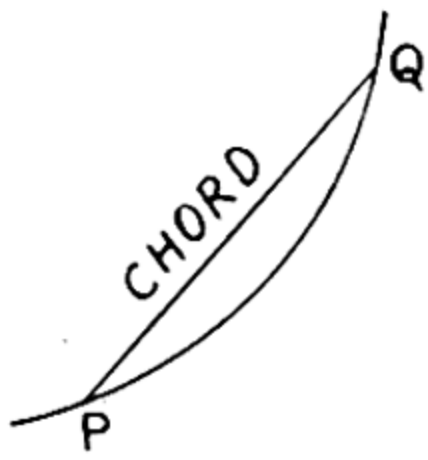
[**Hint.** ∵ ∠ at O=1 rt. ∠, the circle has the join of (3, 0) and (0, 4) as diameter.



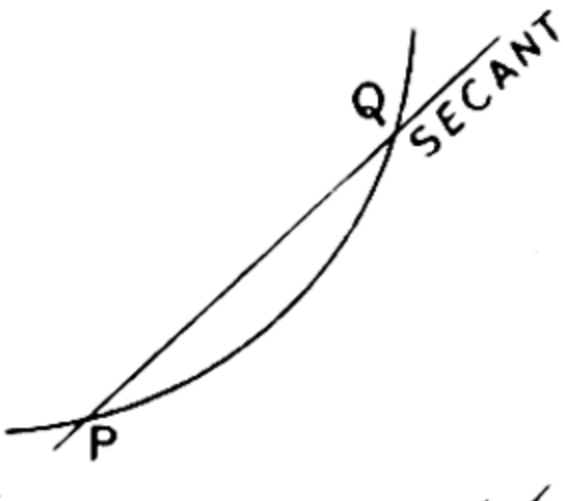
Or thus : Hint. The circle passes thro' (0, 0), (3, 0), (0, 4).]

4. Find the equation to the circle which goes through the origin and cuts off intercepts equal to a and b from the positive parts of the axes. [*P.U. 1952 S*]

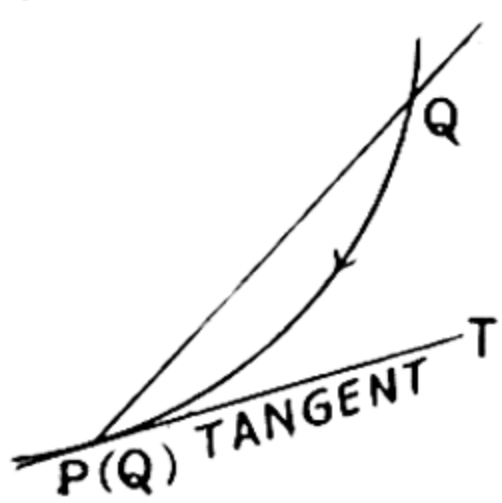
SECTION II
TANGENTS AND NORMALS



61. Def. 1. Chord. Let P and Q be two points on a curve. Join PQ. Then the segment PQ is called a **chord.**



Def. 2. Secant. The chord PQ, produced both ways, is called a **secant.**



Def. 3. Tangent. Let P be a *given* point on a curve and Q *any* other point on the curve. Now let Q move *along* the curve and come to coincide* with P. Then the limiting position PT of the secant PQ is called the **tangent** to the curve at P.

P is called the **point of contact.**

62. Equation of the tangent. To find the equation of the tangent at the point (x_1, y_1) of the circle $x^2+y^2=a^2$.†

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*Strictly speaking, let Q *approach* P.

† [*J. & K. U. 1955*]

[**Method of chord.**] The equation of the circle is $x^2+y^2=a^2$.
Let (x_2, y_2) be any other pt. on the circle.
(*i*) The equation of the line passing thro' (x_1, y_1), (x_2, y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \ldots (1)$$

(*ii*) $\left[\text{To find the value of } \frac{y_2-y_1}{x_2-x_1}.\right]$

$\because$ (x_1, y_1), (x_2, y_2) lie on the circle,
$\therefore$ $x_1^2+y_1^2=a^2$ ...(2)
$x_2^2+y_2^2=a^2$ ...(3)
Subtracting (2) from (3),

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)=0.$$

Factorising, $(x_2-x_1)(x_2+x_1)+(y_2-y_1)(y_2+y_1)=0$,
or $(y_2-y_1)(y_2+y_1)=-(x_2-x_1)(x_2+x_1)$, $\therefore \frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1}{y_2+y_1}$.

(*iii*) Substituting this value of $\frac{y_2-y_1}{x_2-x_1}$ in (1), *the equation of the chord thro'* (x_1, y_1), (x_2, y_2) is $y-y_1=-\frac{x_2+x_1}{y_2+y_1}(x-x_1)$ ...(4)

(*iv*) Putting $x_2=x_1$*, $y_2=y_1$ in (4), the equation of the tangent at (x_1, y_1) is

$$y-y_1=-\frac{2x_1}{2y_1}(x-x_1) \qquad \text{[Cancel 2]}$$

or $yy_1-y_1^2=-xx_1+x_1^2$
or $xx_1+yy_1=x_1^2+y_1^2=a^2$ [$\because$ $x_1^2+y_1^2=a^2$, from (2)]
or $\mathbf{xx_1+yy_1=a^2}$, which is the required equation.

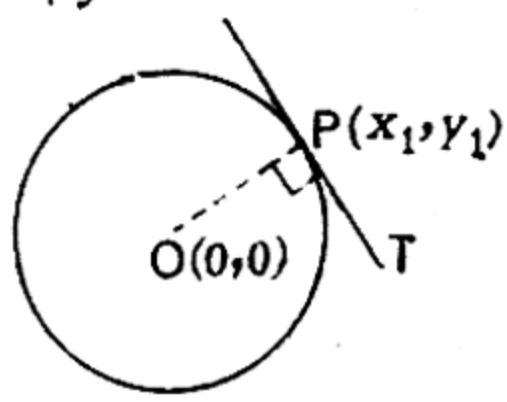
[**N.B.** *The equation of the tangent at the point* (x_1, y_1) *of the circle* $x^2+y^2=a^2$ *is* $\mathbf{xx_1+yy_1=a^2}$.]

Note. Method of chord. Since in this method, first the equation of the *chord* through (x_1, y_1), (x_2, y_2) is found, it is called the **method of chord.**

****Shorter method.** The equation of the circle is $x^2+y^2=a^2$.

The tangent at any point of a circle is perpendicular to the radius through that point.

[From Geometry]



(*i*) The equation of *any* line thro' (x_1, y_1) is

$$y-y_1=m(x-x_1) \quad \ldots(1)$$

(*ii*) If it is $\perp$ to the radius OP

$$\left(\text{slope}=\frac{y_1-0}{x_1-0}=\frac{y_1}{x_1}\right),$$

then $m.\frac{y_1}{x_1}=-1$ [$m_1m_2=-1$]

$\therefore$ $m=-\frac{x_1}{y_1}$.

---

*Strictly speaking, $x_2\to x_1$ (read "x_2 tends to x_1"), $y_2\to y_1$.

(*iii*) Substituting this value of m in (1), the equation of the tangent at (x_1, y_1) is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1) \text{ or } yy_1-y_1^2=-xx_1+x_1^2$$

or $$xx_1+yy_1=x_1^2+y_1^2=a^2$$

[$\because$ $x_1^2+y_1^2=a^2$ as (x_1, y_1) lies on the circle]

or $\mathbf{xx_1+yy_1=a^2}$, which is the required equation.

[**Comparison of the two methods**. The foregoing method is applicable **only to the circle,** *while the first method is applicable to other curves in later chapters as well* (*Arts. 99, 122 and Art. 142, Ex.*).]

63. [**Conic. Def.** *A* **conic** *is a curve whose equation is of the second degree in x and y.*

Rule to find the equation of the tangent at the point (x_1, y_1) of a circle (or conic) :

[**Method of chord.**] Let (x_2, y_2) be any other pt. on the curve.

(*i*) *Write down the equation of the line passing thro'* (x_1, y_1), (x_2, y_2), *viz.*,

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1).$$

(*ii*) [To find the value of $\frac{y_2-y_1}{x_2-x_1}$.]

$\because$ (x_1, y_1), (x_2, y_2) *lie on the curve,* $\therefore$ *substitute these co-ordinates in the equation of the curve, subtract, factorise and find the value of* $\frac{y_2-y_1}{x_2-x_1}$.

(*iii*) *Substitute this value of* $\frac{y_2-y_1}{x_2-x_1}$ *in the equation of step* (*i*), *and thus find the equation of the chord thro'* (x_1, y_1), (x_2, y_2).

(*iv*) *Putting* $x_2=x_1$, $y_2=y_1$ *in the equation of step* (*iii*), *find the equation of the tangent at* (x_1, y_1), *and simplify this equation by means of the condition that* (x_1, y_1) *lies on the curve. The result is the required equation.*]

64. Equation of the tangent to the general circle. To find the equation of the tangent at the point (x_1, y_1) of the circle

$$\mathbf{x^2+y^2+2gx+2fy+c=0.}*$$

[**Method of chord.**] The equation of the circle is

$$x^2+y^2+2gx+2fy+c=0.$$

Let (x_2, y_2) be any other pt. on the circle.

(*i*) The equation of the line passing thro' (x_1, y_1), (x_2, y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \ldots(1)$$

(*ii*) [To find the value of $\frac{y_2-y_1}{x_2-x_1}$.]

$\because$ (x_1, y_1), (x_2, y_2) lie on the circle,

$\therefore$ $x_1^2+y_1^2+2gx_1+2fy_1+c=0 \quad \ldots(2)$

$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \quad \ldots(3)$

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*[*J. & K. U. 1957*]

Subtracting (2) from (3),

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)+2g(x_2-x_1)+2f(y_2-y_1)=0.$$

Factorising,

$$(x_2-x_1)(x_2+x_1)+(y_2-y_1)(y_2+y_1)+2g(x_2-x_1)+2f(y_2-y_1)=0$$

or $(y_2-y_1)(y_2+y_1+2f)+(x_2-x_1)(x_2+x_1+2g)=0$

or $(y_2-y_1)(y_2+y_1+2f)=-(x_2-x_1)(x_2+x_1+2g)$

$$\therefore \quad \frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1+2g}{y_2+y_1+2f}.$$

(*iii*) Substituting this value of $\frac{y_2-y_1}{x_2-x_1}$ in (1), *the equation of the chord thro'* (x_1, y_1), (x_2, y_2) is

$$y-y_1=-\frac{x_2+x_1+2g}{y_2+y_1+2f}(x-x_1) \quad \ldots(4)$$

(*iv*) Putting $x_2=x_1$, $y_2=y_1$ in (4), the equation of the tangent at (x_1, y_1) is

$$y-y_1=-\frac{2x_1+2g}{2y_1+2f}(x-x_1) \qquad [\text{ Cancel 2 }]$$

or $yy_1+fy-y_1^2-fy_1=-xx_1+x_1^2-gx+gx_1$

or $xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1.$

Adding gx_1+fy_1+c *to both sides,* [To make the R.H.S. zero]

$$xx_1+yy_1+gx+fy+gx_1+fy_1+c=x_1^2+y_1^2+2gx_1+2fy_1+c=0$$

[$\because$ $x_1^2+y_1^2+2gx_1+2fy_1+c=0$, from (2)]

or $\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0,}$

which is the required equation.

[**N.B.** *The equation of the tangent at the point* (x_1, y_1) *of the circle*

$$x^2+y^2+2gx+2fy+c=0$$

is $\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.}$]

[**Important. Rule to** *write down* **the equation of the tangent at the point** $(\mathbf{x_1, y_1})$ **of a circle (or conic** in whose equation there is no* term involving the product *xy*, *i.e.*, dealt with in *this* book) : *In the equation of the curve, change* $\mathbf{x^2}$ *to* $\mathbf{xx_1}$, $\mathbf{y^2}$ *to* $\mathbf{yy_1}$, $\mathbf{x}$ *to* $\frac{1}{2}(\mathbf{x+x_1})$, $\mathbf{y}$ *to* $\frac{1}{2}(\mathbf{y+y_1})$.

Important. *If the numerical values of* x_1 *and* y_1 *are given, substitute them in the above result.*]

Example. Find the equation of the tangent to the circle :

(*a*) $\mathbf{x^2+y^2=169}$ **at the point (5, −12) ;** [*P* (*P*). *U.*]

(*b*) $\mathbf{x^2+y^2-4x+2y+3=0}$ **at the point (1, −2).**

(*a*) The equation of the circle is $x^2+y^2=169$.

$\therefore$ the equation of the tangent at (5, −12) is [Art. 64, Rule†]

[Here $x_1=5$, $y_1=-12$]

$$x(5)+y(-12)=169 \text{ or } 5x-12y-169=0.$$

(*b*) The equation of the circle is $x^2+y^2-4x+2y+3=0$.

$\therefore$ the equation of the tangent at (1, −2) is [Art. 64, Rule]

[Here $x_1=1$, $y_1=-2$]

---

*If, however, there *is* a term involving the product *xy*, *change* $\mathbf{xy}$ to $\frac{1}{2}(\mathbf{xy_1+x_1y})$.

†**Explanation.** In the equation of the circle $x^2+y^2=169$, change x^2 to xx_1, *i.e.*, $x(5)$, y^2 to yy_1, *i.e.*, $y(-12)$.

$$x(1)+y(-2)-4.\ \tfrac{1}{2}(x+1)+2.\ \tfrac{1}{2}(y-2)+3=0^*$$

or $$x-2y-2(x+1)+(y-2)+3=0$$

or $$-x-y-1=0 \text{ or } x+y+1=0.$$

[**Check.** The equation of the tangent is satisfied by the co-ordinates of the given pt., thus in (*a*), $5(5)-12(-12)-169=0$ or $0=0$, and in (*b*), $1+(-2)+1=0$ or $0=0$.]

EXAMPLES

1. Write down the equation of the tangent to the circle :

(*i*) $\mathbf{4x^2+4y^2=25}$ **at** $\left(-\frac{3}{2}, -2\right)$,

(*ii*) $\mathbf{2x^2+2y^2+3x-4y+1=0}$ **at** $(-1, 2)$.

2. What is the condition that the circle $x^2+y^2+2gx+2fy+c=0$ should pass through the origin ? If this condition is satisfied, show that the equation of the tangent at the origin is $gx+fy=0$. [*B.H.U. 1948*]

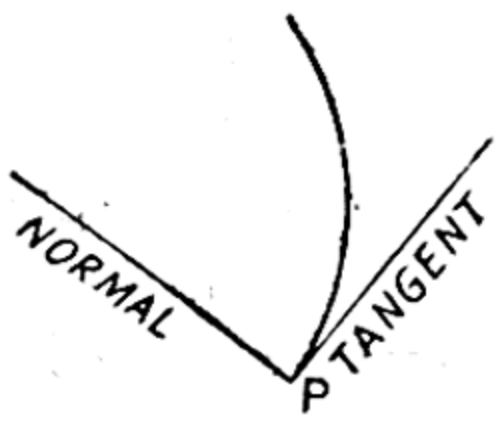
3. Write down the equations of the tangents to the circle $x^2+y^2=10$ at the points whose abscissa is 1. [*P. U.*]

4. Obtain the equation of the tangent at any point (x', y') lying on the circle $x^2+y^2+2gx+2fy+c=0$. [*J. & K. U. 1957*]

Show that the circles $x^2+y^2-4x+6y+8=0$ *and* $x^2+y^2-10x-6y+14=0$ *touch at* $(3, -1)$. [*D.U. 1937*]

[**Hint.** (*ii*) Show that the pt. $(3, -1)$ lies on both the circles, and find the equations of the tangents to the circles at this pt. Show that these equations reduce to the same.]

65. Normal. Def. *The* **normal** *at any point P of a curve is the straight line passing through P and perpendicular to the tangent at P.*



66. Equation of the normal. To find the equation of the normal at the point (x_1, y_1) of the circle $x^2+y^2=a^2$.†

The equation of the circle is $x^2+y^2=a^2$.

(*i*) The equation of the tangent at (x_1, y_1) is
$$xx_1+yy_1=a^2.$$ [Art. 62]

$\therefore$ the slope of the tangent $=-\frac{x_1}{y_1}$.

(*ii*) $\therefore$ the slope of the normal $=\frac{y_1}{x_1}$ [−ve reciprocal]

(*iii*) $\therefore$ the equation of the normal at (x_1, y_1) is

$$y-y_1=\frac{y_1}{x_1}(x-x_1)$$ [$y-y_1=m(x-x_1)$]

***Explanation.** In the equation of the circle $x^2+y^2-4x+2y+3=0$, change x^2 to xx_1, *i.e.*, $x(1)$, y^2 to yy_1, *i.e.*, $y(-2)$, x to $\frac{1}{2}(x+x_1)$, *i.e.*, $\frac{1}{2}(x+1)$, y to $\frac{1}{2}(y+y_1)$, *i.e.*, $\frac{1}{2}(y-2)$.

†[*D.U. 1947*]

or $x_1y-x_1y_1=xy_1-x_1y_1$ or $xy_1-x_1y=0$.

****Shorter method.** *The normal at any point P of a circle is the radius through P.*

The equation of the circle is $x^2+y^2=a^2$.

$\therefore$ the equation of the normal at (x_1, y_1) (1) (2)

[passing thro' the centre (0, 0) and (x_1, y_1)] (0, 0) (x_1, y_1)

is $y-0=\frac{y_1-0}{x_1-0}(x-0)$ $\left[\, y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \,\right]$

or $xy_1-x_1y=0$.

[**Comparison of the two methods.** The foregoing method is applicable **only to the circle,** *while the first method is applicable to other curves in later chapters as well* (Arts. 100, 123 and Art. 142, Ex.).]

Cor. *The normal to a circle at any point passes through the centre.*

Let the equation of the circle be $x^2+y^2=a^2$.

Then the equation of the normal at (x_1, y_1) is $xy_1-x_1y=0$. [Art. 66]

In this equation, substituting the co-ordinates of the centre (0, 0), we get $0\ (y_1)-x_1(0)=0$ or $0=0$, which is true.

$\therefore$ the normal passes thro' the centre.

Note. Important. The *geometrical* properties of a circle (or curve) do not depend on the *form* of the equation of the circle (or curve), therefore we take the *simplest* form of the equation (as here) to prove these properties.

67. [Rule to find the equation of the normal at the point (x_1, y_1) of a circle (or conic) :

(i) Write down the equation of the tangent at (x_1, y_1) [Art. 64, Rule], *and find the slope of the tangent.*

(ii) Find the slope of the normal [*=−ve reciprocal of the slope of the tangent* ($\because m_2=-\frac{1}{m_1}$, Art. 36-I)].

(iii) Write down the equation of the normal at (x_1, y_1) *by using* $y-y_1=m(x-x_1)$. *The result is the required equation.*]

68. Equation of the normal to the general circle. To find the equation of the normal at the point (x_1, y_1) of the circle

$$x^2+y^2+2gx+2fy+c=0.*$$

The equation of the circle is $x^2+y^2+2gx+2fy+c=0$.

(i) The equation of the tangent at (x_1, y_1) is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$ [Art. 64]

or $x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$.

$\therefore$ the slope of the tangent $=-\frac{x_1+g}{y_1+f}$.

$\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y} \text{ (Art. 25, Cor.)}\right]$

(ii) $\therefore$ the slope of the normal $=\frac{y_1+f}{x_1+g}$. [−ve reciprocal]

(iii) $\therefore$ the equation of the normal at (x_1, y_1) is

$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$. $[y-y_1=m(x-x_1)]$

---

*[*C. U. 1949*]

Example. *Find the equation of the normal to the circle :*
(*a*) $x^2+y^2=169$ *at the point* $(5, -12)$;
(*b*) $x^2+y^2-4x+2y+3=0$ *at the point* $(1, -2)$.
(*a*) The equation of the circle is $x^2+y^2=169$.
(*i*) The equation of the tangent at $(5, -12)$ is
$$x(5)+y(-12)=169 \text{ or } 5x-12y-169=0.$$
$$\therefore \text{ the slope of the tangent} = -\frac{5}{-12}=\frac{5}{12}.$$
(*ii*) $\therefore$ the slope of the normal $=-\frac{12}{5}$. [−ve reciprocal]
(*iii*) $\therefore$ the equation of the normal at $(5, -12)$ is
$$y-(-12)=-\frac{12}{5}(x-5) \qquad [y-y_1=m(x-x_1)]$$
or $$y+12=-\frac{12}{5}(x-5)$$
or $$5y+60=-12x+60 \quad \text{or} \quad 12x+5y=0.$$
(*b*) The equation of the circle is $x^2+y^2-4x+2y+3=0$.
(*i*) The equation of the tangent at $(1, -2)$ is
$$x(1)+y(-2)-4.\tfrac{1}{2}(x+1)+2.\tfrac{1}{2}(y-2)+3=0$$
or $$x-2y-2(x+1)+(y-2)+3=0$$
or $$-x-y-1=0 \text{ or } x+y+1=0.$$
$\therefore$ the slope of the tangent $=-1$.
(*ii*) $\therefore$ the slope of the normal $=-\frac{1}{-1}$ [−ve reciprocal]
$$=1.$$
(*iii*) $\therefore$ the equation of the normal at $(1, -2)$ is
$$y-(-2)=1(x-1) \qquad [y-y_1=m(x-x_1)]$$
or $$y+2=1(x-1) \text{ or } x-y-3=0.$$

EXAMPLES

1. *Find the equation of the normal to the circle :*
(*i*) $4x^2+4y^2=25$ *at* $\left(-\frac{3}{2}, -2\right)$;
(*ii*) $2x^2+2y^2+3x-4y+1=0$ *at* $(-1, 2)$.

2. Find the equations of the tangent and normal to the circle :
(*i*) $\mathbf{x^2+y^2=\frac{a^2b^2}{a^2+b^2}}$ **at the point** $\left(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}\right)$.
(*ii*) $\mathbf{x^2+y^2+2gx+2fy=0}$ **at the origin.**

3. Obtain the condition for the straight line $y=mx+c$ to be a normal to the circle whose centre is (a, b) and radius r. [*B.H.U. 1946*]

SECTION III

A CIRCLE AND A LINE

69. Points of intersection. To find the points of intersection of the line $y=mx+c$ with the circle $x^2+y^2=a^2$.

The equation of the line is $y=mx+c$ ... (1)
,, ,, ,, circle is $x^2+y^2=a^2$ ... (2)

(*i*) Substituting the value of y from (1) in (2),

$$x^2+(mx+c)^2=a^2$$

or $$x^2+m^2x^2+2mcx+c^2=a^2$$

or $$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \quad \ldots (3)$$

which is a quadratic in x giving two values of x.

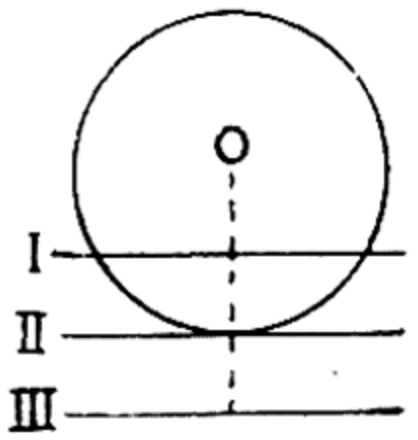
(*ii*) Substituting these values of x one by one in (1), we get the corresponding values of y.

These corresponding values of x and y are the co-ordinates of the required pts. of intersection.

Note 1. Important. To prove that a given line touches a given circle and to find the point of contact. If the roots of the quadratic (3) are *equal*, the line *touches* the circle at the point where x=the equal root. Substituting this value of x in the equation of the line, we get the value of y.

These values of x and y are the co-ordinates of the required point of contact.

****Note 2.** The roots of the quadratic (3) are real, equal, or imaginary according as $4m^2c^2-4(1+m^2)(c^2-a^2)$ is +ve, 0, or −ve, [b^2-4ac is +ve, 0, or −ve], (cancel 4), *i.e.*, according as $m^2c^2-(c^2-a^2+m^2c^2-a^2m^2)$ is +ve, 0, or −ve, *i.e.*, according as $a^2(1+m^2)-c^2$ is +ve, 0, or −ve, *i.e.*, according as $c^2 <, =,$ or $> a^2(1+m^2)$ or $\dfrac{c}{\sqrt{1+m^2}} <, =,$ or $> a$, *i.e.*, the length of the ⊥ from the centre on the line (Fig. I) <, (Fig. II) =, or (Fig. III) >the radius, *i.e.*, according as the line cuts the circle, touches the circle (*i.e.*, cuts it in one point), or does not cut the circle at all. For reasons of generalisation, it is better to say, when the line touches the circle, that it cuts the circle in two co-incident points and to say, when the line does not cut the circle, that it cuts the circle in two imaginary points which cannot be shown in the Fig.



[For, their co-ordinates involve $i(=\sqrt{-1})$.]

[Rule to find the points of intersection of a line and a circle (or conic) :

[*To solve the two equations simultaneously.*]

(*i*) *Substitute the value of y (or the value of x, whichever is simpler) from the equation of the line in the equation of the curve.*

(*ii*) *Solve the resulting quadratic in x.*

(*iii*) *Substitute these values of x one by one in the equation of the* **line,** *and find the corresponding values of y.*

These corresponding values of x and y are the co-ordinates of the required points of intersection.]

EXAMPLES

1. Find the points of intersection of the line $x-2y+10=0$ and the circle $x^2+y^2=25$.

2. Find the points of intersection of the line $y=mx+a$ and the circle $x^2+y^2=a^2$.

3. Find where the line $3x+4y+7=0$ cuts the circle

$$x^2+y^2-4x-6y-12=0.$$ [*C.U.*]

70. Length of the intercept. To find the length of the intercept made by the circle $x^2+y^2=a^2$ on the line $y=mx+c$.*

The equation of the line is $y=mx+c$ ... (1)

,, ,, ,, circle is $x^2+y^2=a^2$ ... (2)

(*i*) Substituting the value of y from (1) in (2),

$$x^2+(mx+c)^2=a^2$$

or $$x^2+m^2x^2+2mcx+c^2=a^2$$

or $$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \quad \ldots (3)$$

which is a quadratic in x. Let x_1, x_2 be the roots.

[To find the value of x_2-x_1.]

From (3), $x_1+x_2=-\dfrac{2mc}{1+m^2}$, [Sum of the roots]

and $$x_1x_2=\frac{c^2-a^2}{1+m^2}.$$ [Product of the roots]

$$\therefore \quad x_2-x_1=\sqrt{(x_2+x_1)^2-4x_2x_1}=\sqrt{\frac{4m^2c^2}{(1+m^2)^2}-\frac{4(c^2-a^2)}{1+m^2}}$$

$$=\sqrt{\frac{4m^2c^2-4(c^2-a^2)(1+m^2)}{(1+m^2)^2}}$$

$$=\frac{2}{1+m^2}\sqrt{m^2c^2-(c^2-a^2)(1+m^2)}$$

$$=\frac{2}{1+m^2}\sqrt{m^2c^2-(c^2+m^2c^2-a^2-a^2m^2)}$$

$$=\frac{2}{1+m^2}\sqrt{-c^2+a^2+a^2m^2}$$

$$=\frac{2}{1+m^2}\sqrt{a^2(1+m^2)-c^2} \quad \ldots (4)$$

(*ii*) [To find the value of y_2-y_1.]

$\because$ (x_1, y_1), (x_2, y_2) lie on the *line*, [*Note this step*]

$\therefore$ $$y_1=mx_1+c \quad \ldots (5)$$

$$y_2=mx_2+c \quad \ldots (6)$$

Subtracting (5) from (6), $y_2-y_1=m(x_2-x_1)$ ... (7)

(*iii*) Now length of the intercept $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$$=\sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2}$$ [From (7)]

$$=(x_2-x_1)\sqrt{1+m^2}$$

$$=\frac{2}{1+m^2}\sqrt{a^2(1+m^2)-c^2}\cdot\sqrt{1+m^2}$$ [From (4)]

$$=\frac{2}{\sqrt{1+m^2}}\sqrt{a^2(1+m^2)-c^2}.$$

[Rule to find the length of the intercept made by a circle (or conic) on the line $y=mx+c$:

(i) Substitute the value of y from the equation of the line in the equation of the curve, and get a quadratic in x. Let x_1, x_2 be the roots.

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*[*B. H. U. 1946*]

[*To find the value of* x_2-x_1.]

Write down the values of x_1+x_2 *and* x_1x_2, *and hence find the value of* x_2-x_1 $(=\sqrt{(x_2+x_1)^2-4x_2x_1})$.

(*ii*) [*To find the value of* y_2-y_1.]

$\because$ (x_1, y_1), (x_2, y_2) *lie on the line,* $\therefore$ *substitute these co-ordinates in the equation of the line, and subtract ; then* $y_2-y_1=m(x_2-x_1)$.

(*iii*) *Now length of the intercept* $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2}=(x_2-x_1)\sqrt{1+m^2}$.

[*From the result of step* (*ii*)]

Substitute from the result of step (*i*).]

[**Note. Short-cut (for the length of the intercept) for numerical examples.** *In a numerical example, if the points of intersection of the line and the curve can be found conveniently, then*

length of the intercept=distance between the points of intersection.]

****Shorter method.**

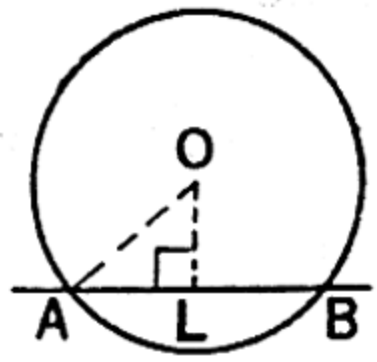
The equation of the line is $y=mx+c$ ... (1)

" " " circle is $x^2+y^2=a^2$ ... (2)

Let the line meet the circle in A, B.

From the centre O draw OL $\perp$ *on AB to bisect it in L.*

[*From Geometry*]



Join OA.

Now $AB=2AL=2\sqrt{OA^2-OL^2}$ ... (3)

But OA (radius)$=a$,

OL=length of the $\perp$ from (0, 0) on the line (1) $mx-y+c=0$,

$$=\frac{c}{\sqrt{1+m^2}}.$$

$$\therefore \text{ from (3), } AB=2\sqrt{a^2-\frac{c^2}{1+m^2}}$$

$$=\frac{2\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}.$$

[**N.B. Intercept formula for the** *circle* **:**

Length of the (chord or) intercept $=2\sqrt{(\text{radius})^2-(\text{central } \perp)^2}$, *where central* $\perp$ *means the* $\perp$ *from the centre on the line.*]

[**Comparison of the two methods. The foregoing method is applicable** *only to the circle* **for which this is, in general, the BEST method** (*specially for the general circle*), *while the first method is applicable to other curves in later chapters as well* [Arts. 105, 125 and Art. 142, Ex. 2.]

Cor. Condition of tangency. To deduce the condition that the line may touch the circle.*

If the line touches the circle, the length of the intercept=0,

i.e., $\frac{2}{\sqrt{1+m^2}}\sqrt{a^2(1+m^2)-c^2}=0$, $\therefore$ $a^2(1+m^2)-c^2$† $=0$

or $c^2=a^2(1+m^2)$, $\therefore$ $c=\pm a\sqrt{1+m^2}$, which is the required condition.

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* [*B.H.U. 1946*]

† *i.e.*, the factor involving *c*.

EXAMPLES

1. Find the co-ordinates of the points where the line $y=2x+1$ cuts the circle $x^2+y^2=2$ and the length of the chord intercepted. [*P. U. 1935S*]

2. Find the length of the chord of the circle $x^2+y^2=r^2$ intercepted on the line $\frac{x}{a}+\frac{y}{b}=1$. [*P. U.*]

[**Hint.** Use the shorter method (Art. 70).]

3. Find the length and** middle point of the chord intercepted by the circle $x^2+y^2=a^2$ from the line $y=px+q$. [*P.U. 1951S*]

[** (*b*) **Shorter method (for the circle only).** *The mid-pt. of the chord of a circle is the pt. of intersection of the given line* ...(1)
and the line thro' the centre $\perp$ *to the given line* ...(2)]

71. Condition of tangency. To find the condition that the line $y=mx+c$ may touch the circle $x^2+y^2=a^2$.*

[**Method of intersections.**]

The equation of the line is $y=mx+c$ ...(1)

,, ,, ,, circle is $x^2+y^2=a^2$ ...(2)

(*i*) Substituting the value of y from (1) in (2),

$$x^2+(mx+c)^2=a^2$$

or $$x^2+m^2x^2+2mcx+c^2=a^2$$

or $$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \quad \ldots(3)$$

which is a quadratic in x.

(*ii*) If the line *touches* the circle, the quadratic (3) has equal roots.

$\therefore\ 4m^2c^2-4(1+m^2)(c^2-a^2)=0$ [$b^2-4ac=0$]

[Cancel 4]

or $$m^2c^2-(c^2-a^2+m^2c^2-a^2m^2)=0$$

or $$m^2c^2-c^2+a^2-m^2c^2+a^2m^2=0 \quad \text{or} \quad c^2=a^2(1+m^2)$$

$\therefore\ \mathbf{c=\pm a\sqrt{1+m^2}},$

which is the required condition.

[**Rule to find the condition that the line $y=mx+c$ may touch a circle (or conic) :**

(*i*) *Substitute the value of y from the equation of the line in the equation of the curve, and get a quadratic in x.*

(*ii*) *If the line touches the curve, the quadratic obtained in step (i) has equal roots. Use $b^2-4ac=0$, and find the value of c. This is the required condition.*]

Shorter method.

The equation of the line is $y=mx+c$ ...(1)

,, ,, ,, circle is $x^2+y^2=a^2$ ...(2)

If the line touches the circle, the length of the $\perp$ *from the centre (0, 0) on the line (1)=the radius* | $mx-y+c=0$

$$\therefore\ \pm\frac{c}{\sqrt{m^2+1}}=a$$ [Complete $\perp$ distance formula]

or $\mathbf{c=\pm a\sqrt{1+m^2}},$

which is the required condition.

---

*[*P. U. 1953S*]

[**Comparison of the two methods. The foregoing method is applicable** *only to the circle* **for which this is, in general, the BEST method** (*specially for the general circle*), *while the first method is applicable to other curves in later chapters as well* (Arts. 106, 126 and Art. 142, Ex. 5).]

Note. Tangent property. Important. In the *shorter method* we have used the

Tangent property. Important. If a line touches a circle, the length of the perpendicular from the centre on the line=the radius.

Note that *for the tangent property we use the* **complete** *perpendicular distance formula.* (See Art. 44-II, Cor., Note.)

Cor. 1. To find the equation of a tangent in terms of its slope. (See Art. 71.) Substituting the value of $c(=\pm a\sqrt{1+m^2})$ in (1), the equation of the tangent is $\mathbf{y=mx\pm a\sqrt{1+m^2}}$.

[**N.B.** *If m is the slope of a tangent to the circle* $x^2+y^2=a^2$, *the equation of the tangent is* $\mathbf{y=mx\pm a\sqrt{1+m^2}}$.]

Cor. 2. Any tangent in the m-form. *The equation of any tangent to the circle* $x^2+y^2=a^2$, *in the m-form, is*

$$\mathbf{y=mx+a\sqrt{1+m^2}}.$$

[Taking +ve sign with the square root in the equation of Cor. 1]

Example 1. Find the equations of the tangents of the circle $x^2+y^2=25$, which are inclined at an angle of 30° to the axis of x. [*A. U. 1946*]

The equation of the circle is $x^2+y^2=25$. [Compare with $x^2+y^2=a^2$]

Here $a^2=25$, $\therefore\ a=5$; $m=\tan 30°=\dfrac{1}{\sqrt{3}}$,

$\therefore$ the equations of the tangents are

$$y=\frac{1}{\sqrt{3}}x\pm 5\sqrt{1+\left(\frac{1}{\sqrt{3}}\right)^2} \qquad [y=mx\pm a\sqrt{1+m^2}\,]$$

$$=\frac{1}{\sqrt{3}}x\pm 5\left(\frac{2}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}x\pm\frac{10}{\sqrt{3}}$$

or $x-\sqrt{3}y+10=0$, $x-\sqrt{3}y-10=0$.

Example 2. Find the equations to the tangents to the circle $x^2+y^2=169$ which are parallel to $5x+12y+21=0$. [*P(P). U.*]

****Method I. [Method of slope.]**

The equation of the circle is $x^2+y^2=169$.

[Compare with $x^2+y^2=a^2$]

Here $a^2=169$, $\therefore\ a=13$.

(*i*) The slope of $5x+12y+21=0$, $=-\dfrac{5}{12}$.

(*ii*) $\therefore$ the slope of the tangents ($\|$ to it)$=-\dfrac{5}{12}$. $[m_1=m_2]$

(*iii*) $\therefore$ the equations of the tangents are

$$y=-\frac{5}{12}x\pm 13\sqrt{1+\left(-\frac{5}{12}\right)^2} \qquad [\,y=mx\pm a\sqrt{1+m^2}\,]$$

$$=-\frac{5}{12}x\pm 13\left(\frac{13}{12}\right)=-\frac{5}{12}x\pm\frac{169}{12}$$

or $5x+12y-169=0,\ 5x+12y+169=0.$

Method II. [**Method of tangent property.**]

The equation of the circle is $x^2+y^2=169$.

(*i*) The equation of *any* line $\parallel$ to $5x+12y+21=0$ is $5x+12y+k=0$ ...(1)

(*ii*) If it touches the circle, *the length of the* $\perp$ *from the centre* $(0, 0)$ *on it=the radius*, $\therefore\ \pm\frac{k}{13}=13$ or $k=\pm 169$.

(*iii*) Substituting this value of k in (1), $5x+12y\pm 169=0$,

or $5x+12y+169=0$, $5x+12y-169=0$, which are the required equations.

[**Comparison of the two methods**. The foregoing method is applicable *only to the circle*, and is the BEST method for the general circle, *while the first method is applicable to other curves in later chapters as well* (Art. 106, Ex. 9 ; Art. 126, Ex. 8 and Art. 142, Ex. 6).]

EXAMPLES

1. Prove that $y=mx+c$ cuts the circle $x^2+y^2=a^2$ in two points. Also find the condition when the line is a tangent to the circle. [*P. U. 1944*]

[**Hint.** (*i*) Proceed as in Art. 69. The equation (3) is a quadratic in x giving two values of x. Substituting these values of x one by one in (1), we get the corresponding values of y.

(*ii*) If the line *touches* the circle, the quadratic (3) has equal roots.]

2. Show that the line $3x+4y+20=0$ touches the circle $x^2+y^2=16$ and find the point of contact. [*P. U. 1932S*]

[**Sol.** The equation of the line is $3x+4y+20=0$ ...(1)

,, ,, ,, circle is $x^2+y^2=16$ ...(2)

Substituting the value of $y\left(=-\frac{3x+20}{4}\right)$ from (1) in (2),

$$x^2+\frac{(3x+20)^2}{16}=16 \text{ or } 16x^2+9x^2+120x+400=256,$$

or $25x^2+120x+144=0$,

$\therefore\ (5x+12)^2=0$, which has equal roots, each $=-\frac{12}{5}$,

$\therefore$ the line touches the circle at the pt. where $x=-\frac{12}{5}$.

Substituting this value of x in $y=-\frac{3x+20}{4}$,

$$y=-\frac{-\frac{36}{5}+20}{4}=-\frac{64}{5(4)}=-\frac{16}{5}$$

$\therefore$ the required pt. of contact is $\left(-\frac{12}{5}, -\frac{16}{5}\right)$.

Note. The above method enables us not only to prove that the line touches the circle, but also to find the *point of contact*.]

3. Prove that $x=7$ and $y=8$ touch the circle $x^2+y^2-4x-6y-12=0$ and find the points of contact. [*C. U. 1955*]

4. Find the condition that (*i*) the line $y=mx+c$ may touch the circle $(x-h)^2+(y-k)^2=a^2$. [*P. U. 1939*]

(*ii*) the straight line $Ax+By+C=0$ may touch the circle $(x-a)^2+(y-b)^2=c^2$. [*J. & K. U. 1952*]

5. Show that the line $y=mx+a\sqrt{1+m^2}$ touches the circle $x^2+y^2=a^2$, whatever the value of m may be, and find the co-ordinates of the point of contact. [*P (P). U. 1956S*]

6. For what value of k will the line $4x+3y+k=0$ touch the circle $2x^2+2y^2=5x$? [*P.U. 1938*]

7. Find the equation of the circle which has its centre at the point (1, −2) and touches the line $x+y+5=0$. [*D. U.*]

8. Point of contact. $Ax+By+C=0$ is a tangent to the circle $x^2+y^2-a^2=0$, find the co-ordinates of the point of contact. [*P. U.*]

[**Sol.** The equation of the circle is $x^2+y^2-a^2=0$.

Let (x_1, y_1) be the required pt. of contact.

The equation of the tangent at (x_1, y_1) is $xx_1+yy_1-a^2=0$...(1)

$\because$ it is the same as the equation of the given line $Ax+By+C=0$...(2)

$\therefore$ comparing coefficients in (1) and (2), $\frac{x_1}{A}=\frac{y_1}{B}=\frac{-a^2}{C}$ [Art. 31]

$\therefore\ x_1=-\frac{A}{C}a^2,\ y_1=-\frac{B}{C}a^2,$

$\therefore$ the required pt. of contact is $\left(-\frac{A}{C}a^2, -\frac{B}{C}a^2\right)$.]

[****Check.** The equation of the tangent at $\left(-\frac{A}{C}a^2, -\frac{B}{C}a^2\right)$

is $x\left(-\frac{A}{C}a^2\right)+y\left(-\frac{B}{C}a^2\right)=a^2$ [Cancel a^2]

or $-Ax-By=C$ or $Ax+By+C=0$,

which is the equation of the given line.]

9. Find the condition that $1x+my+n=0$ may touch the circle $x^2+y^2=a^2$. Assuming the condition to have been satisfied, obtain the co-ordinates of the point of contact. [*J. & K. U. 1956*]

[**Sol.** [**Method of point of contact.**]

The equation of the circle is $x^2+y^3=a^2$.

Let (x_1, y_1) be the pt. of contact.

(*a*) (*i*) The equation of the tangent at (x_1, y_1) is

$xx_1+yy_1=a^2$ or $xx_1+yy_1-a^2=0$ ...(1)

(*ii*) $\because$ it is the same as the equation of the given line

$lx+my+n=0$ ...(2)

$\therefore$ comparing coefficients in (1) and (2), $\frac{x_1}{l}=\frac{y_1}{m}=\frac{-a^2}{n}$ [Art. 31]

$\therefore\ x_1=-\frac{l}{n}a^2,\ y_1=-\frac{m}{n}a^2$ ...(3)

(*iii*) $\because$ (x_1, y_1) lies on the circle, $\therefore\ x_1^2+y_1^2=a^2$

$\therefore$ substituting from (3), $\frac{l^2}{n^2}a^4+\frac{m^2}{n^2}a^4=a^2$ or $a^2l^2+a^2m^2=n^2$,

which is the required condition.

(*b*) From (3), the pt. of contact is $\left(-\frac{l}{n}a^2, -\frac{m}{n}a^2\right)$. [x_1, y_1]

Note. The above method enables us not only to find the condition that the line may touch the circle, but also to find the *point of contact* when the condition is satisfied. It is neater than the method of intersections, and is applicable to other curves in later chapters as well (Art. 106, Ex. 4, Misc. Ex 20, Chap. VI, Art. 126, Ex. 6, and Art. 142, Ex. 9).]

10. Find the condition that the straight line $y=mx+c$ should touch the circle $x^2+y^2=r^2$ and find the point of contact when the condition is satisfied. [*B.U. 1937*]

11. Find the equations of the tangents to the circle $x^2+y^2=4$, which are inclined at an angle of 60° to the axis of x. [*P.U. 1949S*]

12. Find the equations to the tangents to the circle $x^2+y^2=4$

(*i*) **which are parallel to the line $x+2y+3=0$.**

(*ii*) which are parallel to the axis of *y*. [*P(P). U. 1954*]

13. Find the equations to the tangents to the circle

$$x^2+y^2-2x-4y-4=0$$

which are parallel to the line $3x-4y-1=0$.

14. Find the equations to the tangents of the circle

(*i*) **$x^2+y^2=9$ which are perpendicular to $x-y-1=0$.** [*P. U.*]

(*ii*) **$x^2+y^2-2x-4y-4=0$ which are perpendicular to the line $3x-4y-1=0$.** [*P (P). U. 1955*]

SECTION IV

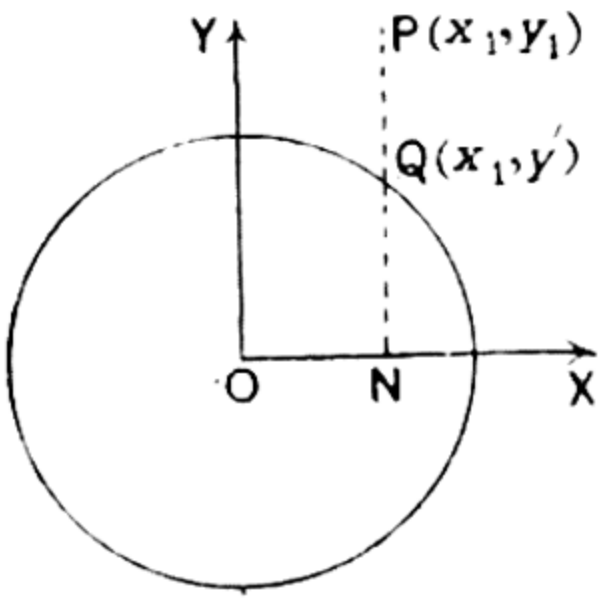
A CIRCLE AND A POINT

72. Test for a point to lie outside, on, or inside a circle. *To show that the quantity $x_1^2+y_1^2-a^2$ is positive, zero, or negative, according as the point (x_1, y_1) lies outside, on, or inside the circle $x^2+y^2=a^2$.*

The equation of the circle is $x^2+y^2=a^2$.

Let P be the pt. (x_1, y_1).

From P draw PN $\perp$ on the *x*-axis to meet the circle in $Q(x_1, y')$.



$\because$ Q lies on the circle

$\therefore$ $x_1^2+y'^2=a^2$

or $x_1^2+y'^2-a^2=0$ ...(1) [R.H.S. zero]

Now $x_1^2+y_1^2-a^2=x_1^2+y_1^2-a^2-0$ ·

[Substitute for 0 from (1)]

$=x_1^2+y_1^2-a^2-(x_1^2+y'^2-a^2)$

$=y_1^2-y'^2=NP^2-NQ^2$ ...(2)

(*i*) If P lies outside the circle (as in the Fig.), NP > NQ,

$\therefore$ from (2)*, $x_1^2+y_1^2-a^2$ is + ve.

(*ii*) If P lies on the circle, NP = NQ,

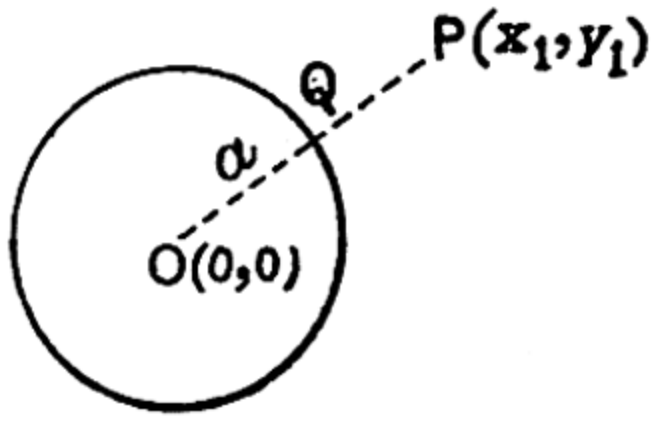
$\therefore$ from (2), $x_1^2+y_1^2-a^2$ is zero.

(*iii*) If P lies inside the circle, NP < NQ,

$\therefore$ from (2), $x_1^2+y_1^2-a^2$ is −ve.

---

* $\because$ R.H.S. is + ve, $\therefore$ L.H.S. is also +ve.



**Shorter method. The equation of the circle is $x^2+y^2=a^2$.

Let P be the pt. (x_1, y_1) and O(0, 0) the centre of the circle. Join OP to meet the circle in Q.

Then $OP=\sqrt{(x_1-0)^2+(y_1-0)^2}$

$=\sqrt{x_1^2+y_1^2}$,

and OQ (radius)$=a$.

(*i*) If P lies outside the circle (as in the Fig.),

$OP > OQ$, *i.e.*, $\sqrt{x_1^2+y_1^2} > a$

or $x_1^2+y_1^2 > a^2$, $\therefore$ $x_1^2+y_1^2-a^2$ is +ve.

(*ii*) If P lies on the circle, $OP=OQ$, *i.e.*, $\sqrt{x_1^2+y_1^2}=a$

or $x_1^2+y_1^2=a^2$, $\therefore$ $x_1^2+y_1^2-a^2$ is zero.

(*iii*) If P lies inside the circle, $OP < OQ$, *i.e.*, $\sqrt{x_1^2+y_1^2} < a$

or $x_1^2+y_1^2 < a^2$, $\therefore$ $x_1^2+y_1^2-a^2$ is −ve.

[**Comparison of the two methods.** *The foregoing method is applicable* **only to the circle (and is the BEST method for the** *general* **circle**), *while the first method is applicable to other curves in later chapters as well* (Arts. 107, 127 and 143.]

[**Rule to find whether a point lies outside, on, or inside a circle (or conic but not hyperbola) (equation in the standard form*) :**

In the L.H.S. of the equation of the curve (*R.H.S. being zero*), *substitute the co-ordinates of the point ; if the result is positive, the point lies outside the curve, if zero, it lies on the curve, and if negative, it lies inside the curve.*]

EXAMPLE

Find whether the point (6, 9) lies inside or outside the circle

$x^2+y^2-100=0$. [*P.U. 1934 S*]

73. Two tangents from a point. *To show that from any point two tangents can be drawn to a circle*†, *and they are real, coincident, or imaginary, according as the point is outside, on, or inside the circle.*

Let the equation of the circle be $x^2+y^2=a^2$, and let the pt. be (x_1, y_1).

(*i*) The equation of *any* tangent to the circle, in the *m*-form, is

$y=mx+a\sqrt{1+m^2}$. [Art. 71, Cor. 2]

(*ii*) If it passes thro' (x_1, y_1), then $y_1=mx_1+a\sqrt{1+m^2}$

or, transposing, $y_1-mx_1=a\sqrt{1+m^2}$

or, squaring, $y_1^2-2mx_1y_1+m^2x_1^2=a^2+a^2m^2$

or $m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-a^2)=0$ ... (1)

which is a quadratic in m giving two values of m.

$\therefore$ from (x_1, y_1) two tangents can be drawn to the circle.

(*iii*) The tangents are real, coincident, or imaginary, according as the roots of the quadratic (1) are real, equal, or imaginary,

i.e., according as $4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)$ is +ve, 0, or −ve

[b^2-4ac is +ve, 0, or −ve]

*In the case of the *general* circle, coeffs. of x^2 and y^2 are each=1.

†[*P.U. 1953S*]

or, according as $x_1^2y_1^2-(x_1^2y_1^2-a^2x_1^2-a^2y_1^2+a^4)$ is +ve, 0, or −ve,
i.e., according as $a^2x_1^2+a^2y_1^2-a^4$ is +ve, 0, or −ve
or, dividing thro' out by a^2, [which is +ve]
according as $x_1^2+y_1^2-a^2$ is +ve, 0, or −ve
i.e., according as the pt. (x_1, y_1) lies outside, on, or inside the circle.
[Art. 72]

Note. We know from Geometry that, when the point lies outside the circle, two tangents can be drawn, and when it lies on the circle, only one tangent can be drawn, and when it lies inside the circle, no tangent can be drawn. But for reasons of generalisation, it is better to say, when the point lies on the circle, that two coincident tangents can be drawn from it and to say, when the point lies inside the circle, that two imaginary tangents can be drawn from it. Thus from any point two (real, coincident, or imaginary) tangents can be drawn to the circle.

Example. Tangents are drawn from the point (4, −2) to the circle $x^2+y^2=10$. Find their equations. [*B. U.*]

Method I. The equation of the circle is $x^2+y^2=10$.
[Compare with $x^2+y^2=a^2$]

Here $a^2=10$, $\therefore\ a=\sqrt{10}$.

(*i*) The equation of *any* tangent to the circle, in the *m*-form, is
$y=mx+\sqrt{10}\sqrt{1+m^2}$ ...(1) [$y=mx+a\sqrt{1+m^2}$ (Art. 71, Cor. 2)]

(*ii*) If it passes thro' (4, −2), then $-2=m(4)+\sqrt{10}\sqrt{1+m^2}$
or, transposing, $-2-4m=\sqrt{10}\sqrt{1+m^2}$
or, squaring, $4+16m+16m^2=10+10m^2$
or $6m^2+16m-6=0$ or $3m^2+8m-3=0$

$$\therefore\quad m=\frac{-8\pm\sqrt{64+36}}{6}=\frac{-8\pm10}{6}=\frac{2}{6},\ -\frac{18}{6}$$
$$=\tfrac{1}{3},\ -3.$$

(*iii*) $\therefore$ the equations of the tangents from (4, −2) are
(I) $y+2=\frac{1}{3}(x-4)$ [$\mathbf{y-y_1=m(x-x_1)}$ (*Note this step*)]
or $3y+6=x-4$ or $x-3y-10=0$.
(II) $y+2=-3(x-4)=-3x+12$
or $3x+y-10=0$.

[**Caution. Important.** When the slopes of the tangents have been found, find their equations by using $y-y_1=m(x-x_1)$ [and *not* by substituting the values of m in $y=mx+\sqrt{10}\sqrt{1+m^2}$ ...(1) as is sometimes done, even in the text books.

For, if we substitute in (1) the value of m found in step (*ii*). *e.g.*, $m=\frac{1}{3}$, we get

$$y=\tfrac{1}{3}x+\sqrt{10}\sqrt{1+\frac{1}{9}}=\tfrac{1}{3}x+\sqrt{10}\,\frac{\sqrt{10}}{3}=\tfrac{1}{3}x+\frac{10}{3}$$

or $x-3y+10=0$, as the equation of the tangent which is wrong, ∵ it does not pass thro' the given pt. (4, −2).

****Note.** The mistake has arisen because there is an ambiguity of sign (±) with the square root in (1). If, however, we take the −ve sign with the square root, we get $y=\frac{1}{3}x-\sqrt{10}\sqrt{1+\frac{1}{9}}=\frac{1}{3}x-\sqrt{10}\,\frac{\sqrt{10}}{3}$

$$=\tfrac{1}{3}x-\frac{10}{3}$$

or $x-3y-10=0$, as the equation of the tangent, which is right.]

[**Rule to find the equations of the tangents drawn from a given point to a circle (or conic) (equation in the standard form) :**

(*i*) *Write down the equation of* **any** *tangent to the curve, in the m-form.*

(*ii*) *If it passes thro' the given pt., substitute the co-ordinates of that pt., and solve the resulting quadratic in m.*

(*iii*) *Write down the equations of the tangents from* (x_1, y_1) *by using* $y-y_1=m(x-x_1)$.]

Method II. [Method of tangent property.]

The equation of the circle is $x^2+y^2=10$.

(*i*) The equation of *any* line thro' $(4, -2)$ is

$$y+2=m(x-4) \quad \ldots(1) \qquad [y-y_1=m(x-x_1)]$$

or $mx-y-4m-2=0$.

(*ii*) If it touches the circle $x^2+y^2=10$,

the length of the $\perp$ from the centre $(0, 0)$ on it=the radius

$$\therefore \pm\frac{-4m-2}{\sqrt{m^2+1}}=\sqrt{10} \text{ or, squaring, } 16m^2+16m+4=10m^2+10$$

or $6m^2+16m-6=0$ or $3m^2+8m-3=0$

$$\therefore \quad m=\frac{-8\pm\sqrt{64+36}}{6}=\frac{-8\pm10}{6}=\frac{2}{6}, -\frac{18}{6}$$

$$=\tfrac{1}{3}, -3.$$

(*iii*) Substituting these values of m in (1),

(I) $y+2=\frac{1}{3}(x-4)$ or $3y+6=x-4$ or $x-3y-10=0$;

(II) $y+2=-3(x-4)=-3x+12$ or $3x+y-10=0$.

$\therefore$ $x-3y-10=0$, $3x+y-10=0$,

which are the required equations.

[**Comparison of the two methods.** The foregoing method is applicable *only to the circle* (and is the BEST method for the *general* circle), *while the first method is applicable to other curves in later chapters as well* (Art. 108, solved Ex. ; Art. 128, Ex. 1 and Art. 143, Ex. 3).]

EXAMPLE

Find the equations of the tangents drawn from the point (4, 2) to the circle $x^2+y^2=4$. State the co-ordinates of their points of contact.

[*P. U. 1934*]

74. Rule to find the locus of a point (x_1, y_1) when an equation between its co-ordinates is given :

In the given equation, change (x_1, y_1) *to* (x, y). *The result is the equation of the required locus.*

EXAMPLE

If $ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0$, find the locus of (x_1, y_1).

[**Sol.** In the equation $ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0$ ...(1), changing (x_1, y_1) to (x, y), $ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(2) which is the equation of the required locus.

[For, equation (2) is satisfied by the co-ordinates of the pt. (x_1, y_1). (From 1)]

75. Director circle. To find the locus of the point of intersection of two perpendicular tangents to a circle.*

Let the equation of the circle be $x^2+y^2=a^2$.

(*i*) The equation of *any* tangent to the circle, in the *m*-form, is

$$y=mx+a\sqrt{1+m^2}. \qquad \text{[Art. 71, Cor. 2]}$$

(*ii*) If it passes thro' (x_1,y_1), then $y_1=mx_1+a\sqrt{1+m^2}$

or, transposing, $y_1-mx_1=a\sqrt{1+m^2}$

or, squaring, $y_1^2-2mx_1y_1+m^2x_1^2=a^2+a^2m^2$

or $m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-a^2)=0$ ...(1)

which is a quadratic in m. Let m_1, m_2 be the roots.

(*iii*) If the tangents are $\perp$, $m_1m_2=-1$.

$\therefore$ from (1), $\dfrac{y_1^2-a^2}{x_1^2-a^2}=-1$,

or $y_1^2-a^2=-x_1^2+a^2$ or $x_1^2+y_1^2=2a^2$.

$\therefore$ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y) (Art. 74)]

$x^2+y^2=2a^2$, which is a circle.

Note. This circle is called the **director circle** of the given circle.

[Rule to find the locus of the point of intersection of two perpendicular tangents to a circle (or conic) (equation in the standard form) :

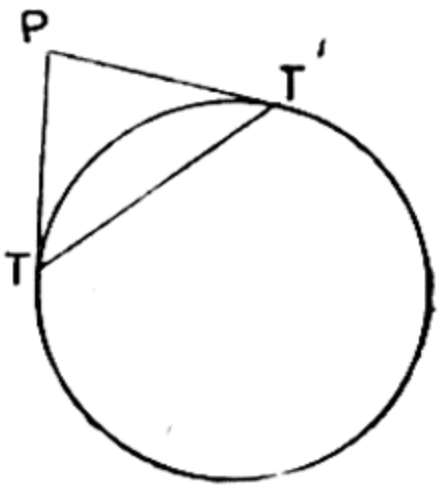
(*i*) *Write down the equation of* **any** *tangent to the curve, in the m-form.*

(*ii*) *If it passes thro'* (x_1, y_1), *substitute these co-ordinates, and write the resulting equation as a quadratic in m. Let* m_1, m_2 *be the roots.*

(*iii*) *If the tangents are* $\perp$, $m_1m_2=-1$. *Substitute from the product of the roots of the quadratic obtained in step* (*ii*), *and change* (x_1, y_1) *to* (x, y). *The result is the equation of the required locus.*]

Chord of contact

76. Chord of contact. Def. *Let PT, PT′ be the two tangents drawn from a point P to a circle* (*or conic*), *then the straight line TT′ joining the points of contact is called the* **chord of contact** *of tangents from P to the circle* (*or conic*).



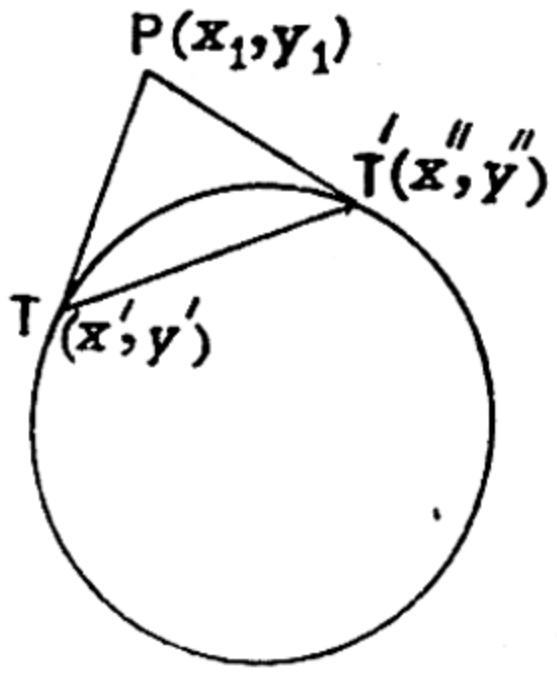
77. Equation of the chord of contact. To find the equation of the chord of contact of tangents from (x_1, y_1) to the circle $x^2+y^2=a^2$.†

*[P. U. 1953S]

† [*M. B. 1952*]

The equation of the circle is $x^2+y^2=a^2$.

Let P be the pt. (x_1, y_1) and T (x', y'), T' (x'', y''), the pts. of contact of the tangents from P to the circle.



(*i*) Then the equations of the tangents at T, T' are

$$xx'+yy'=a^2 \quad ...(1)$$
$$xx''+yy''=a^2 \quad ...(2)$$

(*ii*) ∵ both these tangents pass thro' P,

$$\therefore \quad x_1x'+y_1y'=a^2 \quad ...(3)$$
$$x_1x''+y_1y''=a^2 \quad ...(4)$$

(*iii*) ∴ from (3) and (4), T (x', y'), T' (x'', y'') lie on the line $x_1x+y_1y=a^2$,* ***i.e.***, $\mathbf{xx_1+yy_1=a^2}$, which is, therefore, the required equation of the chord of contact. [Def. (Art. 76)]

[**N.B.** *The equation of the chord of contact of tangents from* (x_1, y_1) *to the circle* $x^2+y^2=a^2$ is $\mathbf{xx_1+yy_1=a^2}$.]

[**Note.** If the point (x_1, y_1) lies *inside* the circle, the reasoning of the foregoing Art. shows that the line joining the imaginary points of contact of the two imaginary tangents from (x_1, y_1) to the circle is $xx_1+yy_1=a^2$, which is always real. Thus we have a *real* line joining the imaginary points of contact of the two imaginary tangents—a result, strange but true !]

EXAMPLES

1. Find the equation of the chord of contact of tangents drawn from the point (h, k) to the circle $x^2+y^2=r^2$. [*B. U. 1938*]

2. Find the points of contact of the tangents to the circle $x^2+y^2=25$ that pass through the point (7, 1) and write down the equations of the tangents. [*D. U. 1940*]

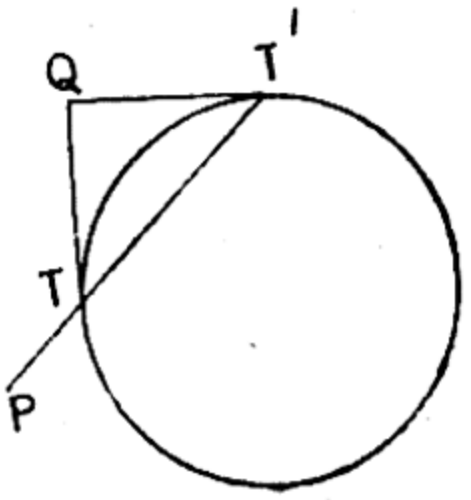
3. Equation of the chord of contact of tangents to the general circle. Show that the equation of the chord of contact of tangents from (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.}$$

[**Hint.** Proceed as in Art. 77, using the equation of the tangent found in Art. 64.]

Pole and polar

78. Pole and polar†. Defs. *Let PTT' be any straight line drawn through a point P to meet a circle (or conic) in T, T', and let the tangents at T, T' meet in Q ; then the locus of Q is called the* **polar** *of P with respect to the circle (or conic) ; P is called the* **pole** *of the locus.*



79. Equation of the polar. To find the equation of the polar of the point (x_1, y_1) with respect to the circle $x^2+y^2=a^2$.‡

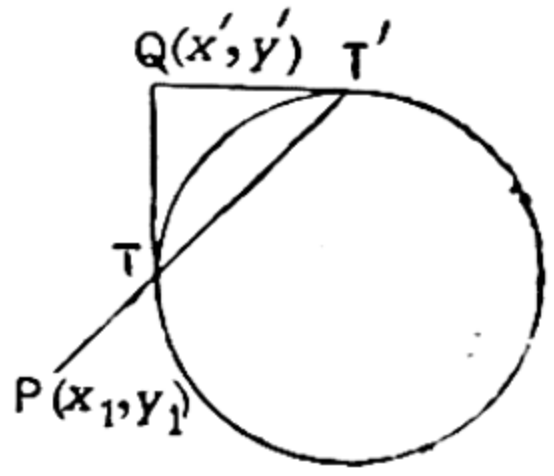
***How to write down this equation.** In the equations (3) and (4), change (x', y') and (x'', y'') to (x, y).

†[*P.U. 1943*]

‡[*Bar. U. 1953*]

The equation of the circle is $x^2+y^2=a^2$.

Let P be the pt. (x_1, y_1) and PTT′ *any* line thro' P to meet the circle in T, T′. Let the tangents at T, T′ meet in Q (x', y').



(*i*) The equation of TT′, the chord of contact of tangents from Q, is

$$xx'+yy'=a^2. \qquad \text{(Art. 77)}$$

(*ii*) ∵ it passes thro' P,

$\therefore\ x_1x'+y_1y'=a^2.$

(*iii*) $\therefore$ the locus of $Q(x', y')$ is [changing (x', y') to (x, y)]

$$x_1x+y_1y=a^2, \textit{ i.e., } \mathbf{xx_1+yy_1=a^2},$$

which [is the required equation of the polar of P [Def. (Art. 78)], and is clearly a st. line.

[**N.B.** *The equation of the polar of* (x_1, y_1) *w. r. t. the circle* $x^2+y^2=a^2$ *is* $\mathbf{xx_1+yy_1=a^2}$.

Notice that *the letters "w. r. t." stand for "with respect to"*.]

[**Aid to memory.** The *form* of the equation of the *tangent* to a circle at (x_1, y_1) is exactly the same as that of the *chord of contact* of tangents from (x_1, y_1), and of the *polar* of (x_1, y_1). *Hence we have only to remember the Rule of Art. 64 for all the three.* As will be seen later, the same is true for a conic.]

Note. Two important particular cases of the polar.

(*i*) *When the point P is outside the circle* (as in the Fig. of Art. 79), *its polar is the same as the chord of contact of tangents from P.*

(Remember)

[For, the equations of Arts. 79 and 77 are the same.]

(*ii*) *When P is on the circle, its polar is the same as the tangent at P.* [For, the equations of Arts. 79 and 62 are the same.]

As will be seen later, these results are true of a *conic*.

EXAMPLES

1. Equation of the polar w. r. t. the general circle. Show that the polar of the point (x_1, y_1) with respect to the circle $x^2+y^2+2gx+2fy+c=0$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$. [*C. U. 1950*]

[**Hint.** Proceed as in Art. 79, using the equation of the chord of contact found in Ex. 3, Art. 77.]

2. Define 'pole and polar' w. r. t. a circle. [*B.U. 1942*]

Find the polar of the point $(-4, 6)$ with respect to the circle $x^2+y^2=36$.

[**Sol.** (*b*) The equation of the circle is $x^2+y^2=36$.

$\therefore$ the equation of the polar of $(-4, 6)$ is

[by the Rule of writing down the equation of the tangent]

[Here $x_1=-4$, $y_1=6$]

$x(-4)+y(6)=36$ or $-2x+3y=18$ or $2x-3y+18=0$.]

3. Find the polar of the point :

(*i*) **$(1, -2)$ with respect to the circle $2x^2+2y^2=3$.**

(*ii*) **$(-1, 2)$ with respect to the circle $x^2+y^2-3x-4y+5=0$.**

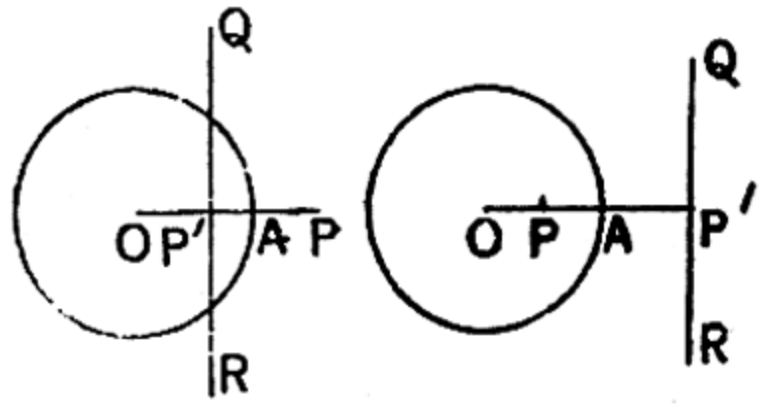
4. Find the polar of the point :

(*i*) **(−a, b) with respect to the circle**
$$x^2+y^2-2ax+2by+a^2-b^2=0.$$
(*ii*) **$(1, -\frac{1}{2})$ with respect to the circle $3x^2+3y^2-4x+5y-6=0$.**

5. Prove that the polar of the point (p, q) with respect to the circle $x^2+y^2=a^2$ touches $(x-h)^2+(y-k)^2=r^2$, if
$$(p^2+q^2)r^2=(a^2-ph-qk)^2.$$ [*P.U. 1948*]

6. Give a geometrical construction for the polar of a point with respect to a circle.

Let P be the given pt., and O the centre of the circle.



Construction. Join OP and on it (produced, if necessary) take a pt. P′ such that OP.OP′=(radius)². Thro' P′ draw a line QR ⊥ to OP ; this is the required polar.

[**Proof.** (*i*) Let the equation of the circle be $x^2+y^2=a^2$, and let P be the pt. (x_1, y_1). The equation of the polar of P is $xx_1+yy_1=a^2$.
$$\text{Its slope}=-\frac{x_1}{y_1}.$$
The slope of OP [passing thro' (0, 0) and (x_1, y_1)] $=\frac{y_1}{x_1}$.

The product of the slopes $=-\frac{x_1}{y_1}\cdot\frac{y_1}{x_1}=-1$; $(m_1m_2=-1)$

∴ the polar of P is ⊥ to OP ...(1)

(*ii*) Let the polar of P meet OP in P′.

Then OP′=length of the ⊥ from (0, 0) on the polar
$$xx_1+yy_1-a^2=0,$$ [From (1)]
$$=\frac{a^2}{\sqrt{x_1^2+y_1^2}};$$
$$\therefore\quad \mathbf{OP.OP'}=\sqrt{x_1^2+y_1^2}\cdot\frac{a^2}{\sqrt{x_1^2+y_1^2}}=\mathbf{a^2}.\ ]$$

7. AB is the polar of a point P (x_1, y_1) with respect to the circle $x^2+y^2=a^2$, centre O. ON is perpendicular to AB. Show that
(*i*) ON.OP$=a^2$.
(*ii*) OP is perpendicular to AB. [*Bar. U. 1953*]

8. Salmon's theorem. The distances of two points from the centre of a circle are proportional to the distances of each from the polar of the other. [*C.U. 1953*]

[**Proof.** Let the equation of the circle be $x^2+y^2=a^2$, and let the two pts. be P(x_1, y_1), Q(x_2, y_2)

Then $\dfrac{\text{distance of P from the centre}}{\text{distance of Q from the centre}}$
$$=\frac{\sqrt{(x_1-0)^2+(y_1-0)^2}}{\sqrt{(x_2-0)^2+(y_2-0)^2}}=\frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}}\quad\ldots(1)$$
The equations of the polars of P, Q are
$$xx_1+yy_1=a^2,$$
$$xx_2+yy_2=a^2.$$

$$\therefore \quad \frac{\text{distance of P from the polar of Q}}{\text{distance of Q from the polar of P}} \quad \begin{vmatrix} xx_2+yy_2-a^2=0 \\ xx_1+yy_1-a^2=0 \end{vmatrix}$$

$$= \frac{\dfrac{x_1x_2+y_1y_2-a^2}{\sqrt{x_2^2+y_2^2}}}{\dfrac{x_2x_1+y_2y_1-a^2}{\sqrt{x_1^2+y_1^2}}} = \frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}} \quad \ldots(2)$$

From (1) and (2),

$$\frac{\text{distance of P from the centre}}{\text{distance of Q from the centre}} = \frac{\text{distance of P from the polar of Q}}{\text{distance of Q from the polar of P}}.$$]

80. Pole of a given line. To find the pole of the line $Ax+By+C=0$ with respect to the circle $x^2+y^2=a^2$.*

The equation of the circle is $x^2+y^2=a^2$.

Let (x_1, y_1) be the required pole.

(*i*) The equation of the polar of (x_1, y_1) is

$$xx_1+yy_1=a^2 \text{ or } xx_1+yy_1-a^2=0 \quad \ldots(1)$$

(*ii*) $\because$ it is the same as the equation of the given line

$$Ax+By+C=0 \quad \ldots(2)$$

(*iii*) $\therefore$ comparing coefficients in (1) and (2), [Art. 31]

$$\frac{x_1}{A} = \frac{y_1}{B} = \frac{-a^2}{C}$$

$$\therefore \quad x_1 = -\frac{A}{C}a^2, \quad y_1 = -\frac{B}{C}a^2$$

$\therefore$ the required pole is $\left(-\frac{A}{C}a^2, -\frac{B}{C}a^2\right)$.

[Rule to find the pole of a given line with respect to a given circle (or conic) :

Let (x_1, y_1) be the required pole.

(*i*) *Write down the equation of the polar of* (x_1, y_1).

(*ii*) *It is the same as the equation of the given line.*

(*iii*) *Compare coefficients in the two equations, and find the values of* x_1, y_1. *These are the co-ordinates of the required pole.*

Note. *A similar Rule is applicable to find the point of contact of a given tangent.* (*Cf.* Art. 71, Ex. 8.)]

Example. Find the pole of the straight line $2x-y=6$ with respect to the circle $5x^2+5y^2=9$. [*A. U. 1947*]

The equation of the circle is $5x^2+5y^2=9$.

Let (x_1, y_1) be the required pole.

(*i*) The equation of the polar of (x_1, y_1) is $5xx_1+5yy_1=9$

or $$5xx_1+5yy_1-9=0 \quad \ldots(1)$$

(*ii*) $\because$ it is the same as the equation of the given line

$$2x-y-6=0 \quad \ldots(2)$$

(*iii*) $\therefore$ comparing coefficients in (1) and (2),

$$\frac{5x_1}{2} = \frac{5y_1}{-1} = \frac{-9}{-6} = \frac{3}{2}$$

---

*[*Bar. U. 1953*]

$\therefore \quad x_1=\frac{3}{5}, y_1=-\frac{3}{10}, \quad \therefore$ the required pole is $\left(\frac{3}{5}, -\frac{3}{10}\right)$.

[**Check. The equation of the polar of $\left(\frac{3}{5}, -\frac{3}{10}\right)$ is

$$5x\left(\frac{3}{5}\right)+5y\left(-\frac{3}{10}\right)=9 \qquad \text{[Cancel 3]}$$

or $$x-\frac{y}{2}=3 \quad \text{or} \quad 2x-y=6,$$

which is the equation of the given line.]

EXAMPLES

1. Find the pole of the straight line $4x-y=5$ with respect to the circle $x^2+y^2=25$. [*B.H.U.*]

****2. Find the pole of the straight line $4x+12y-9=0$ with respect to the circle $3x^2+3y^2+4x-6y+4=0$.**

[**Sol.** The equation of the circle is $3x^2+3y^2+4x-6y+4=0$.

Let (x_1, y_1) be the required pole.

(*i*) The equation of the polar of (x_1, y_1) is

$$3xx_1+3yy_1+4\,.\,\tfrac{1}{2}(x+x_1)-6\,.\,\tfrac{1}{2}(y+y_1)+4=0$$

or $$3xx_1+3yy_1+2(x+x_1)-3(y+y_1)+4=0$$

or $$x(3x_1+2)+y(3y_1-3)+(2x_1-3y_1+4)=0 \qquad \ldots(1)$$

(*ii*) $\because$ it is the same as the equation of the given line

$$4x+12y-9=0 \qquad \ldots(2)$$

(*iii*) $\therefore$ comparing coefficients in (1) and (2),

$$\frac{3x_1+2}{4}=\frac{3y_1-3}{12}=\frac{2x_1-3y_1+4}{-9} \quad \text{or} \quad \frac{3x_1+2}{4}=\frac{y_1-1}{4}=\frac{2x_1-3y_1+4}{-9}.$$

From the first two members, $3x_1+2=y_1-1$ or $3x_1-y_1+3=0$ ...(3)

From the last two members, $-9y_1+9=8x_1-12y_1+16$

or $$8x_1-3y_1+7=0 \qquad \ldots(4)$$

Solving (3) and (4) by cross-multiplication,

$$\begin{matrix} x_1 & y_1 & 1 \\ -1 & 3 & 3 & -1 \\ -3 & 7 & 8 & -3 \end{matrix}$$

$$\therefore \quad \frac{x_1}{-7+9}=\frac{y_1}{24-21}=\frac{1}{-9+8} \quad \text{or} \quad \frac{x_1}{2}=\frac{y_1}{3}=\frac{1}{-1}$$

$\therefore \quad x_1=-2, y_1=-3, \quad \therefore$ the required pole is $(-2, -3)$.]

[****Check.** The equation of the polar of $(-2, -3)$ is

$$3x(-2)+3y(-3)+4.\ \tfrac{1}{2}(x-2)-6\,.\,\tfrac{1}{2}(y-3)+4=0$$

or $$-6x-9y+2x-4-3y+9+4=0 \quad \text{or} \quad -4x-12y+9=0$$

or $4x+12y-9=0$, which is the equation of the given line.]

****3. Find the pole of the straight line**

(*i*) $x+2y+12=0$ with respect to the circle $x^2+y^2+3x-4y-1=0$.

(*ii*) $7x-6y=3$ with respect to the circle $2x^2+2y^2-3x+4y-1=0$.

4. Prove that if the pole of a straight line with respect to the circle $x^2+y^2=a^2$ lie on the circle $x^2+y^2=9a^2$, the line is a tangent to the circle

$$x^2+y^2=\frac{a^2}{9}. \qquad \text{[} A.U.\ 1932 \text{]}$$

Properties of pole and polar

81. Reciprocal property. *If the polar of a point P with respect to a circle passes through a point Q, then the polar of Q passes through P.**

Let the equation of the circle be $x^2+y^2=a^2$. Let P, Q be the pts. (x_1, y_1), (x_2, y_2). Then the equation of the polar of P is $xx_1+yy_1=a^2$.

If it passes thro' Q, then $x_2x_1+y_2y_1=a^2$ ...(1)

Now the equation of the polar of Q is $xx_2+yy_2=a^2$ ...(2)

Substituting the co-ordinates of P in (2), we get $x_1x_2+y_1y_2=a^2$, which is true from (1). ∴ the polar of Q passes thro' P.

Conjugate points. Def. Two points are said to be **conjugate** with respect to a circle (or conic) when the polar of each passes through the other.

EXAMPLE

If the polars of two points P and Q with respect to a circle meet in R, then R is the pole of the line PQ.

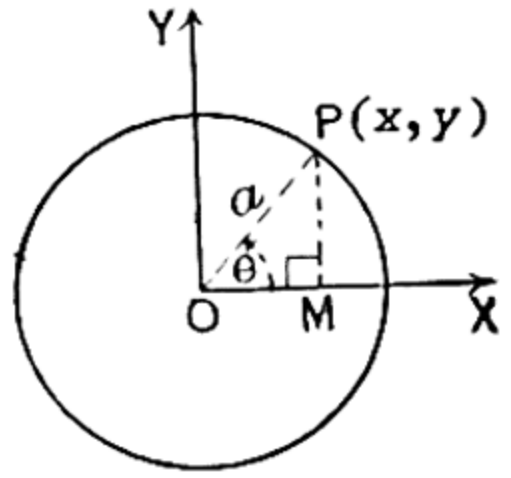
[**Proof.** ∵ the polar of P passes thro' R, ∴ the polar of R passes thro' P (Art. 81). Similarly the polar of R passes thro' Q.

∴ the polar of R is PQ, *i.e.*, R is the pole of PQ.]

Use of parameter

82. To express the co-ordinates of any point on the circle $x^2+y^2=a^2$ in terms of one variable (called the *parameter*).

Let P (x, y) be *any* pt. on the circle. From P draw PM ⊥ on the x-axis, and join OP.



Let $\angle XOP=\theta$.

Then $x=OM$ [But $\frac{OM}{OP}=\cos\theta$,

$\therefore\ OM=OP\cos\theta$]

$=a\cos\theta$.

Again $y=MP$ [But $\frac{MP}{OP}=\sin\theta$, $\therefore\ MP=OP\sin\theta$]

$=a\sin\theta$.

$\therefore$ **$x=a\cos\theta$, $y=a\sin\theta$,** which are the required co-ordinates.

****Another method.** The equation of the circle is $x^2+y^2=a^2$

or $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$ ...(1) [R.H.S.=1]

Also $\cos^2\theta+\sin^2\theta=1$ ...(2)

$\therefore$ comparing (1) and (2), $\frac{x^2}{a^2}=\cos^2\theta$, $\frac{y^2}{a^2}=\sin^2\theta$

or $\frac{x}{a}=\cos\theta$, $\frac{y}{a}=\sin\theta$,

or **$x=a\cos\theta$, $y=a\sin\theta$,** which are the required co-ordinates.

---

*[*A.B. 1951*]

[**Comparison of the two methods.** The foregoing method does not give the geometrical meaning of θ, *i.e.*, $\angle XOP=\theta$.]

Cor. Any point on the circle. *Any point on the circle* $x^2+y^2=a^2$ *is* **(a cos θ, a sin θ).**

Note 1. Since the position of the point P (a cos θ, a sin θ) is known when the value of θ is given, hence the

Def. *The point whose co-ordinates are* (a *cos* θ, a *sin* θ) *is, for shortness, called the* **"point θ".**

Note 2. Def. A variable in terms of which the co-ordinates of *any* point on a circle (or conic) are expressed, is called a **parameter,** and the co-ordinates are called the **parametric co-ordinates.**

****Note 3. What is the advantage of using parametric co-ordinates ?**

The advantage of using parametric co-ordinates (a cos θ, a sin θ) is that we have to give the value of *one* quantity θ to express the position (co-ordinates) of a point P on the circle, whereas in ordinary co-ordinates we have to give the values of *two* quantities x_1, y_1 (which are, of course, connected by the relation $x_1^2+y_1^2=a^2$).

EXAMPLES

1. Prove that a chord of a circle is at right angles to the diameter through its middle point. [*P. U. 1946*]

2. The chord of contact of tangents drawn from any point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$. Show that a, b, c are in G.P. [*P. U. 1950*]

3. Prove that $x=a$ cos θ, $y=a$ sin θ together represent a circle of radius a. [*C. U. 1954*]

83. *To find the equation of the straight line joining two points,* θ_1 *and* θ_2, *on the circle* $x^2+y^2=a^2$.

The equation of the circle is $x^2+y^2=a^2$.

The co-ordinates of the pts. θ_1, θ_2

(1) (2)

are $(a\cos\theta_1, a\sin\theta_1)$, $(a\cos\theta_2, a\sin\theta_2)$. [Def. (Art. 82)]

$\therefore$ the equation of the line passing thro' them is

$$y-a\sin\theta_1=\frac{a\sin\theta_2-a\sin\theta_1}{a\cos\theta_2-a\cos\theta_1}(x-a\cos\theta_1) \quad \left[\, y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\,\right]$$

[Cancel a from the num. and denom.]

$$=\frac{2\cos\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}}{2\sin\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}}(x-a\cos\theta_1)$$

[Cancel $2\sin\frac{\theta_1-\theta_2}{2}$ from the num. and denom.]

$$=-\frac{\cos\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_1+\theta_2}{2}}(x-a\cos\theta_1)$$

$$\text{or} \quad y \sin \frac{\theta_1+\theta_2}{2} - a \sin\theta_1 \sin \frac{\theta_1+\theta_2}{2}$$

$$= -x \cos \frac{\theta_1+\theta_2}{2} + a \cos\theta_1 \cos \frac{\theta_1+\theta_2}{2}$$

$$\text{or} \quad x \cos \frac{\theta_1+\theta_2}{2} + y \sin \frac{\theta_1+\theta_2}{2}$$

$$= a\left(\cos\theta_1 \cos \frac{\theta_1+\theta_2}{2} + \sin\theta_1 \sin \frac{\theta_1+\theta_2}{2}\right)$$

$$= a \cos\left(\theta_1 - \frac{\theta_1+\theta_2}{2}\right) = a \cos \frac{\theta_1-\theta_2}{2}$$

$$\therefore \quad x \cos \frac{\theta_1+\theta_2}{2} + y \sin \frac{\theta_1+\theta_2}{2} = a \cos \frac{\theta_1-\theta_2}{2} \quad \ldots(1)$$

which is the required equation.

Cor. *To deduce the equation of the tangent at the point* θ_1.

Putting $\theta_2=\theta_1$ in (1), the equation of the tangent at the pt. θ_1 is

$$x \cos\theta_1 + y \sin\theta_1 = a \cos 0 = a$$

or

$$x \cos\theta_1 + y \sin\theta_1 = a.$$

EXAMPLE

1. Find the equations of the tangent and normal at the point (a cos θ, a sin θ) of the circle $x^2+y^2=a^2$.

[**Sol.** The equation of the circle is $x^2+y^2=a^2$.

(i) The equation of the tangent at $(a \cos\theta, a \sin\theta)$ is [Art. 64, Rule]

[Here $x_1 = a\cos\theta$, $y_1 = a\sin\theta$]

$$x(a\cos\theta) + y(a\sin\theta) = a^2 \qquad \text{[Cancel } a\text{]}$$

or

$$x\cos\theta + y\sin\theta = a.$$

(ii) (*i*) The equation of the tangent at $(a \cos\theta, a \sin\theta)$ is

$x \cos\theta + y \sin\theta = a$. [From part (i)]

$$\therefore \text{ the slope of the tangent} = -\frac{\cos\theta}{\sin\theta}.$$

(*ii*) $\therefore$ the slope of the normal $= \dfrac{\sin\theta}{\cos\theta}$. [−ve reciprocal]

(*iii*) $\therefore$ the equation of the normal at $(a \cos\theta, a \sin\theta)$ is

$$y - a\sin\theta = \frac{\sin\theta}{\cos\theta}(x - a\cos\theta)$$

or

$$y\cos\theta - a\sin\theta\cos\theta = x\sin\theta - a\sin\theta\cos\theta$$

or

$$x\sin\theta - y\cos\theta = 0.]$$

****2.** Express the co-ordinates of any point on the circle $(x-h)^2+(y-k)^2=a^2$ in terms of one variable.

[**Sol.** The equation of the circle is $(x-h)^2+(y-k)^2=a^2$.

Put $x-h=x'$, $y-k=y'$, then the equation becomes $x'^2+y'^2=a^2$, and the co-ordinates of *any* pt. on this circle are $x'=a\cos\theta$, $y'=a\sin\theta$ [Art. 82].

$\therefore$ $x-h=a\cos\theta$, $y-k=a\sin\theta$ or $x=h+a\cos\theta$, $y=k+a\sin\theta$, which are the required co-ordinates.]

**3. Show that the equation of the straight line joining the points θ and θ' on the circle $(x-h)^2+(y-k)^2=a^2$ is

$(x-h)\cos\frac{\theta+\theta'}{2}+(y-k)\sin\frac{\theta+\theta'}{2}=a\cos\frac{\theta-\theta'}{2}$, and deduce that the equation of the tangent at the point θ is $(x-h)\cos\theta+(y-k)\sin\theta=a$.

[**Hint.** Put $x-h=x'$, $y-k=y'$, so that the equation of the circle becomes $x'^2+y'^2=a^2$. Proceed as in Art. 83 using x', y' for x, y, and find the equation of the line joining the pts. θ and θ', as

$$x'\cos\frac{\theta+\theta'}{2}+y'\sin\frac{\theta+\theta'}{2}=a\cos\frac{\theta-\theta'}{2}.$$

But $x'=x-h$, $y'=y-k$. For the tangent, put $\theta'=\theta$.]

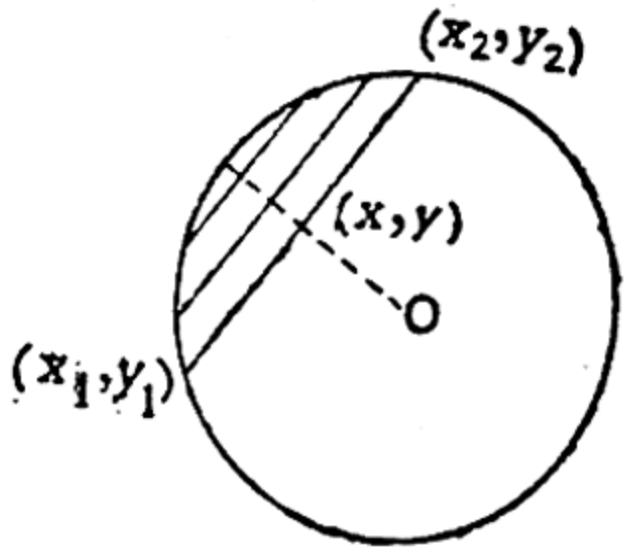
SECTION V

GEOMETRICAL PROPERTIES

84. Diameter property. *To find the locus of the middle points of a system of parallel chords of a circle.*

Let the equation of the circle be

$$x^2+y^2=a^2.$$



(*i*) Let (x_1, y_1), (x_2, y_2) be the ends of *any* chord of the system and (x, y) its mid-pt.

Then $x=\frac{x_1+x_2}{2}$, $y=\frac{y_1+y_2}{2}$ ...(1)

Let m be the slope of the $\parallel$ chords.

Then $m=\frac{y_2-y_1}{x_2-x_1}$ ...(2)

(*ii*) [To find the value of $\frac{y_2-y_1}{x_2-x_1}$.]

$\because$ (x_1, y_1), (x_2, y_2) lie on the circle,

$\therefore$ $x_1^2+y_1^2=a^2$ ...(3)

$x_2^2+y_2^2=a^2$ ...(4)

Subtracting (3) from (4),

$(x_2^2-x_1^2)+(y_2^2-y_1^2)=0$.

Factorising, $(x_2-x_1)(x_2+x_1)+(y_2-y_1)(y_2+y_1)=0$,

or $(y_2-y_1)(y_2+y_1)=-(x_2-x_1)(x_2+x_1)$

$\therefore$ $\frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1}{y_2+y_1}$ ...(5)

(*iii*) Substituting the values of $\frac{y_2-y_1}{x_2-x_1}(=m)$, $x_2+x_1(=2x)$, $y_2+y_1(=2y)$ from (2) and (1) in (5),

$$m=-\frac{2x}{2y} \qquad [\text{ Cancel 2 }]$$

or $$\mathbf{y=-\frac{1}{m}x},$$ which is the required locus.

This is a st. line passing thro' the origin, *i.e.*, thro' the centre of the circle and perpendicular to the chords.

[$\because$ its slope $= -\frac{1}{m}$

($-$ ve reciprocal of the slope of the chords)]

Diameter. Def. *The locus of the middle points of a system of parallel chords of a circle (or conic) is called a* **diameter.***

[**N.B.** *The equation of the diameter of the circle* $x^2+y^2=a^2$, *bisecting parallel chords of slope* m, *is* $\mathbf{y=-\frac{1}{m}x}$.]

[Rule to find the locus of the middle points of a system of parallel chords of a circle (or conic) :

(*i*) *Let* (x_1, y_1), (x_2, y_2) *be the ends of* **any** *chord of the system and* (x, y) *its mid-pt. Then* $x=\frac{x_1+x_2}{2}$, $y=\frac{y_1+y_2}{2}$ ...(1)

Let m be the slope of the $\|$ *chords.*

Then $m=\frac{y_2-y_1}{x_2-x_1}$ ...(2)

(*ii*) [*To find the value of* $\frac{y_2-y_1}{x_2-x_1}$.]

$\because$ (x_1, y_1), (x_2, y_2) *lie on the curve, substitute these co-ordinates in the equation of the curve, subtract, factorise, and find the value of* $\frac{y_2-y_1}{x_2-x_1}$.

(*iii*) *Substitute in the result of step* (*ii*), *the values of* $\frac{y_2-y_1}{x_2-x_1}(=m)$, $x_2+x_1(=2x)$, $y_2+y_1(=2y)$ *from* (*2*) *and* (*1*).

The result is the required locus.]

EXAMPLES

****1. Equation of a chord in terms of its mid-point. The equation of the chord of the circle $x^2+y^2=a^2$, whose mid-point is (x_1, y_1), is $xx_1+yy_1=x_1^2+y_1^2$.** [*P.U. 1953*]

[**Sol. (*i*) The equation of *any* chord thro' (x_1, y_1) is

$$y-y_1=m(x-x_1) \quad ...(1)$$

(*ii*) $\because$ the mid-pt. (x_1, y_1) of the chord lies on the diameter

$$y=-\frac{1}{m}x$$

$$\therefore\ y_1=-\frac{1}{m}x_1 \text{ or } m=-\frac{x_1}{y_1}.$$

(*iii*) Substituting this value of m in (1),

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

or $$yy_1-y_1^2=-xx_1+x_1^2 \text{ or } xx_1+yy_1=x_1^2+y_1^2 \quad ...(1)$$

Shorter method. (*i*) The equation of *any* chord thro' (x_1, y_1) is

$$y-y_1=m(x-x_1) \quad ...(1)$$

---

*This definition agrees with the definition of a diameter of a circle given in Geometry, *viz.*, *a chord of a circle passing through the centre.*

(*ii*) If it is $\perp$ to the line passing thro' the centre (0, 0) and the mid-pt. (x_1, y_1) $\left[\text{ slope}=\frac{y_1}{x_1}\right]$, then $m\left(\frac{y_1}{x_1}\right)=-1$; $\quad [\ m_1m_2=-1\ ]$

$$\therefore \quad m=-\frac{x_1}{y_1}.$$

(*iii*) Substituting this value of m in (1), $y-y_1=-\frac{x_1}{y_1}(x-x_1)$

or $\quad yy_1-y_1^2=-xx_1+x_1^2$ or $xx_1+yy_1=x_1^2+y_1^2$ ... (1)

[**Comparison of the two methods.** The foregoing method is applicable *only to the circle* while the first method is applicable to other curves in later chapters as well.]

Note. Abridged notation. Adding ($-a^2$) to both sides of (1)*, it becomes

$$xx_1+yy_1-a^2=x_1^2+y_1^2-a^2 \text{ ... (2)}$$

If $S=x^2+y^2-a^2$, so that $S=0$ is the equation of the circle,

$S_1=x_1^2+y_1^2-a^2$, so that S_1 is the result of substituting the co-ordinates of the pt. (x_1, y_1) in S,

$T=xx_1+yy_1-a^2$, so that $T=0$ is the equation of the tangent at (x_1, y_1), then from (2), the equation of the chord can be written as

$$\mathbf{T=S_1}.$$

[**N.B.** *The equation of the chord of the circle (or conic) $S=0$, whose mid-point is (x_1, y_1), is* $\mathbf{T=S_1}$,
where $T=0$ is the equation of the tangent at (x_1, y_1), and S_1 is the result of substituting the co-ordinates of the point (x_1, y_1) in S.]

2. Find the equation to the chord of the circle $x^2+y^2=9$ which is bisected at the point (−1, 2). [*A.U.*]

[**Sol.** The equation of the circle is

$$x^2+y^2=9 \text{ or } x^2+y^2-9=0 \qquad [\text{ Form } S=0\ ]$$

[Here $x_1=-1, y_1=2$]

$\therefore \quad T=x(-1)+y(2)-9=-x+2y-9,$

$S_1=(-1)^2+(2)^2-9=-4,$

$\therefore$ the equation of the chord is

$$-x+2y-9=-4 \qquad [\ T=S_1\ ]$$

or $\quad x-2y+5=0.$]

[**Check.** The equation of the chord is satisfied by the co-ordinates of the mid-point thus :

$$-1-2(2)+5=0 \text{ or } 0=0.\ ]$$

3. Find the equation to that chord of the circle $x^2+y^2=16$ which is bisected at the point (−2, 3), and its pole with respect to the circle. [*C.U.*]

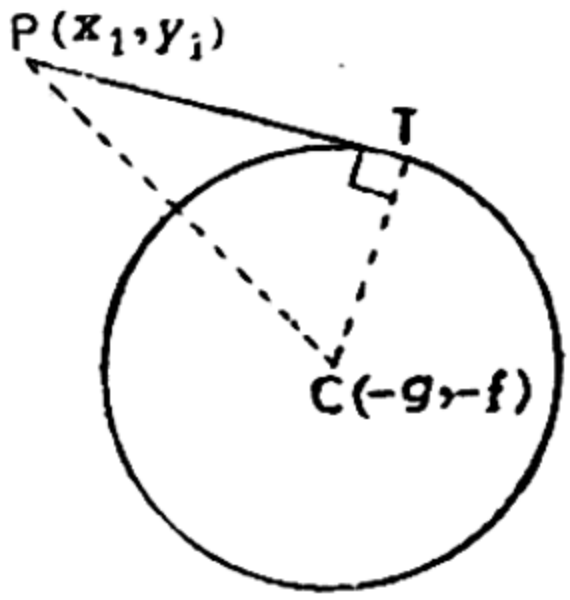
Length of the tangent

85. To find the length of the tangent from the point (x_1, y_1) to the circle $x^2+y^2+2gx+2fy+c=0$.†

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*Why this step. In order to get $x_1^2+y_1^2-a^2$, *i.e.*, S_1 on the R.H.S. we add $(-a^2)$ to both sides.

†[*P. U. 1952S*]



The equation of the circle is
$$x^2+y^2+2gx+2fy+c=0.$$
Let P be the pt. (x_1, y_1), PT a tangent from P and C $(-g, -f)$ the centre of the circle. Join CT and CP.

Then from the rt. $\angle$d $\triangle$ CTP,
$$PT^2=CP^2-CT^2 \quad ...(1)$$
But $CP^2=[\ x_1-(-g)\ ]^2+[\ y_1-(-f)\ ]^2$
[Distance formula]
$$=(x_1+g)^2+(\ y_1+f\ )^2,$$
and $CT^2=(\text{radius})^2=g^2+f^2-c,$
$\therefore$ from (1),
$$PT^2=(x_1+g)^2+(y_1+f\ )^2-(g^2+f^2-c)$$
$$=x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2-g^2-f^2+c$$
$$\therefore\ PT=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}.$$

[Rule to find the length of the tangent from a point to a circle whose equation is given :

In the L.H.S. of the equation of the circle (coefficients of x^2, y^2 being each=1, R.H.S. being zero), substitute the co-ordinates of the point, and take the square root of the result. The result gives the length of the tangent.]

Example. Find the length of the tangent drawn from the point (1, −1) to the circle $2x^2+2y^2-6x-3y+17=0$.

The equation of the circle is $2x^2+2y^2-6x-3y+17=0$

or, *dividing thro' out by 2,* $x^2+y^2-3x-\frac{3}{2}y+\frac{17}{2}=0.$

[Coeffs. of x^2, $y^2=1$ (*Note this step*)]

$\therefore$ length of the tangent from (1, −1)
$$=\sqrt{(1)^2+(-1)^2-3(1)-\frac{3}{2}(-1)+\frac{17}{2}}$$
[Art. 85, Rule*]
$$=\sqrt{1+1-3+\frac{3}{2}+\frac{17}{2}}=\sqrt{9}=3.$$

EXAMPLES

1. Find the length of the tangent drawn from the point (x_1, y_1) to the circle $x^2+y^2=a^2$. [*B.U. 1945*]

Find the length of the tangent drawn from the point (−1, 2) to the circle $3x^2+3y^2=11$.

2. Find the length of the tangent drawn

(*i*) from the point (2, 5) to the circle $x^2+y^2-2x-3y-1=0$. [*D.U. 1946*]

***Explanation.** In the L. H. S., $x^2+y^2-3x-\frac{3}{2}y+\frac{17}{2}$, substitute the co-ordinates of (1, −1), thus getting $(1)^2+(-1)^2-3(1)-\frac{3}{2}(-1)+\frac{17}{2}$, and take the square root of the result, thus getting
$$\sqrt{(1)^2+(-1)^2-3(1)-\frac{3}{2}(-1)+\frac{17}{2}}.$$

(*ii*) from the point (x', y') to the circle $a(x^2+y^2)+2gx+2fy+c=0$. [*C.U.*]

3. The length of the tangent drawn from (f, g) to the circle $x^2+y^2=6$ is twice the length of the tangent from (f, g) to the circle $x^2+y^2+3x+3y=0$, show that $f^2+g^2+4f+4g+2=0$. [*C.U.*]

4. If the length of the tangent from any point to the circle $x^2+y^2+2x=0$ be three times the length of the tangent from the same point to the circle $x^2+y^2=4$, show that the point must be on the circle

$$4x^2+4y^2-x-18=0.$$ [*P. U. 1942S*]

5. Find the point from which the tangents to the three circles $x^2+y^2-16x+60=0$, $x^2+y^2-12x+27=0$, and $x^2+y^2-16x-12y+84=0$ are equal in length. [*A. U. 1943*]

Find also this length.

SECTION VI

TWO OR MORE CIRCLES

Radical axis of two circles.

86. Radical axis. Def.* The *radical axis* of two circles is the locus of a point which moves so that the tangents from it to the two circles are equal.

87. Equation of the radical axis. To find the equation of the radical axis of the two circles

$$x^2+y^2+2g_1x+2f_1y+c_1=0,$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0.$$*

The equations of the circles are $x^2+y^2+2g_1x+2f_1y+c_1=0$ ...(1)

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \ldots(2)$$

(*i*) Let (x, y) be *any* pt. on the radical axis.

(*ii*) Then length of the tangent from (x, y) to (1)
=length of the tangent from (x, y) to (2). [Def. (Art. 86)]

(*iii*) $\therefore \sqrt{x^2+y^2+2g_1x+2f_1y+c_1}=\sqrt{x^2+y^2+2g_2x+2f_2y+c_2}$ [Art. 85]

or, squaring, $x^2+y^2+2g_1x+2f_1y+c_1=x^2+y^2+2g_2x+2f_2y+c_2$

or $$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0,$$

which is the required equation of the radical axis, and is clearly a st. line. [Art. 24]

Cor. 1. *The radical axis of two circles passes through their points of intersection.*

Proof. The co-ordinates of the pts. which satisfy both (1) and (2), also satisfy

$$(x^2+y^2+2g_1x+2f_1y+c_1)-(x^2+y^2+2g_2x+2f_2y+c_2)=0 \quad \ldots(3)$$

[$\because$ substituting from (1) and (2) in (3), we get $0-0=0$, which is true]

i.e., $$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0,$$

which is the equation of the radical axis.

$\therefore$ the pts. of intersection of the circles (1) and (2) lie on the radical axis.

---

*[*P. U. 1955S*]

Cor. 2. Important. (*i*) *The radical axis of two intersecting circles is their common chord.*

(*ii*) *The radical axis of two touching circles is their common tangent at the point of contact.*

[Rule to write down the equation of the radical axis of two circles :

Write the equation of each circle so that the coefficients of x^2, y^2 are each=1 on the L.H.S., R H.S. being zero, and subtract one equation from the other. The resulting equation is the equation of the radical axis.]

Example 1. Find the radical axis of the circles

$$\mathbf{x^2+y^2+x-y-2=0 \text{ and } 3x^2+3y^2+4x-12=0.}$$

The equations of the circles are

$x^2+y^2+x-y-2=0$ ...(1) [Coeffs. of x^2, y^2 already=1]

$3x^2+3y^2+4x-12=0$

or, *dividing thro' out by 3,* $x^2+y^2+\frac{4}{3}x-4=0$ ...(2)

[Coeffs. of x^2, y^2=1 (*Note this step*)]

Subtracting (2) from (1), $-\frac{1}{3}x-y+2=0$ or $x+3y-6=0$, which is the required equation of the radical axis.

Example 2. Touching circles. Show that the circles

$$\mathbf{x^2+y^2+2x+2y+1=0 \text{ and } x^2+y^2-4x-6y-3=0}$$

touch each other. Find the point of contact. [*Pesh. U. 1951*]

Method I. [Method of radical axis.]

The equations of the circles are $x^2+y^2+2x+2y+1=0$ ...(1)

$x^2+y^2-4x-6y-3=0$ ...(2)

(*i*) Subtracting (2) from (1),

the equation of the radical axis is $6x+8y+4=0$

or $3x+4y+2=0$ ...(3)

(*ii*) Substituting the value of $y\left(=-\frac{3x+2}{4}\right)$ from (3) in (1),

$$x^2+\frac{(3x+2)^2}{16}+2x-\frac{(3x+2)}{2}+1=0$$

or $16x^2+9x^2+12x+4+32x-24x-16+16=0$ or $25x^2+20x+4=0$,

$\therefore$ $(5x+2)^2=0$, which has equal roots, each$=-\frac{2}{5}$.

$\therefore$ the radical axis touches the circle (1) [and $\therefore$ (2)] at the pt. where $x=-\frac{2}{5}$.

Substituting this value of x in $y=-\frac{3x+2}{4}$,

$$y=-\frac{-\frac{6}{5}+2}{4}=-\frac{4}{5(4)}=-\frac{1}{5}$$

$\therefore$ the pt. of contact is $\left(-\frac{2}{5},-\frac{1}{5}\right)$.

Note. To find the equation of the common tangent. *The common tangent being the radical axis,* its equation is $3x+4y+2=0$. [From (3)]

Method II. [Method of centre and radius.]

The equations of the circles are $x^2+y^2+2x+2y+1=0$ ...(1)
$x^2+y^2-4x-6y-3=0$ ...(2)

Here $g_1=\frac{1}{2}(2)$ [$\frac{1}{2}$ coeff. of x] $=1$,	$g_2=\frac{1}{2}(-4)=-2$,
$f_1=\frac{1}{2}(2)$ [$\frac{1}{2}$ coeff. of y] $=1$,	$f_2=\frac{1}{2}(-6)=-3$,
$c_1=1$. [Constant term]	$c_2=-3$.

(*i*) $\therefore$ the centres of the circles are $(-1, -1)$, $(2, 3)$

and their radii are $\sqrt{(1)^2+(1)^2-1}=\sqrt{1}=1$,

$\sqrt{(-2)^2+(-3)^2-(-3)}=\sqrt{4+9+3}=\sqrt{16}=4$.

(*ii*) $\therefore$ distance between the centres $=\sqrt{[2-(-1)]^2+[3-(-1)^2}$

$=\sqrt{(3)^2+(4)^2}=\sqrt{9+16}=\sqrt{25}$

$=5$,

also sum of the radii $=1+4=5$

$\therefore$ the circles touch externally.

The pt. of contact divides the join of the centres $(-1, -1)$ and $(2, 3)$ internally in the ratio of the radii $1 : 4$

$\therefore$ its co-ordinates are $x=\dfrac{1(2)+4(-1)}{1+4}=-\dfrac{2}{5}$,

$y=\dfrac{1(3)+4(-1)}{1+4}=-\dfrac{1}{5}$.

$\therefore$ the pt. of contact is $\left(-\dfrac{2}{5}, -\dfrac{1}{5}\right)$.

Note. To find the equation of the common tangent. *The common tangent being the tangent to the circle (1) at the point of contact* $\left(-\frac{2}{5}, -\frac{1}{5}\right)$, its equation is

$$x\left(-\frac{2}{5}\right)+y\left(-\frac{1}{5}\right)+2\cdot\frac{1}{2}\left(x-\frac{2}{5}\right)+2\cdot\frac{1}{2}\left(y-\frac{1}{5}\right)+1=0$$

or $\dfrac{3}{5}x+\dfrac{4}{5}y+\dfrac{2}{5}=0$ or $3x+4y+2=0$.

[Two methods to prove that two circles touch :

Method I. [Method of radical axis.]

(*i*) *Find the equation of the radical axis of the two circles.*

(*ii*) *Show that the radical axis touches one of the two circles.*

Note 1. *The point of contact of the two circles is the point of contact of the radical axis and one of the two circles.*

Note 2. *The common tangent is the radical axis.*

Method II. [Method of centre and radius.]

(*i*) *Find the centres and radii of the circles.*

(*ii*) *Show that the distance between the centres is equal to the sum or the difference of the radii (according as the circles touch externally or internally).*

Note 1. *The point of contact divides the join of the centres in the ratio of the radii internally in the first case and externally in the second.*

Note 2. *The common tangent is the tangent to one of the two circles at the point of contact.*]

EXAMPLES

1. Find the radical axis of the circles :

(*i*) $x^2+y^2=4$ and $x^2+y^2-2x+3y=0$.

(*ii*) $x^2+y^2+4x-4y+7=0$ and $x^2+y^2-6x+2y-3=0$. [*C.U.*]

(*iii*) $x^2+y^2-bx+by+c=0$ and $ax^2+ay^2-a^2x+b^2y=0$.

2. Find the radical axis and the length of the common chord of the circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+bx+ay+c=0$.

[*P.U. 1957S*]

3. Find the length of the common chord of the circles given by the equations $x^2+y^2+2x+3y+1=0$, and $x^2+y^2+4x+3y+2=0$.

[*A. U. 1947*]

4. Points of intersection of two circles. Find the points of intersection of the circles $x^2+y^2-2x+2y=23$ and $x^2+y^2=25$.

[Rule to find the points of intersection of two circles :

(*i*) *Find the equation of the radical axis of the two circles.*

(*ii*) *Find the points of intersection of this radical axis and one of the two circles. These are the points of intersection of the two circles.*]

5. Touching circles. Prove that the two circles $x^2+y^2+2ax+c=0$, $x^2+y^2+2by+c=0$ touch if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$. [*J. & K. U. 1957*]

6. Show that the circles $x^2+y^2-6x-6y+10=0$, and $x^2+y^2=2$ touch and find the point of contact. [*P. U.*]

7. Prove that the circles

$$x^2+y^2-2x-4y+1=0 \text{ and } x^2+y^2+4x+4y-1=0$$

touch each other and find the equation of the common tangent. [*P.U.*]

8. Show that the circles $x^2+y^2-2x=0$ and $x^2+y^2+6x-6y+2=0$ touch each other. Find the co-ordinates of the point of contact and equation of the tangent there.

Is the contact external or internal ? [*P. U. 1954S*]

88. The radical axis of two circles is perpendicular to the line joining their centres.*

Proof. Let the equations of the circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \ldots(1)$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \ldots(2)$$

Then the equation of their radical axis is [Subtracting (2) from (1)]

$2x(g_1-g_2)+2y(f_1-f_2)+(c_1-c_2)=0$ [Art. 87].

$\therefore$ m_1=slope of the radical axis $=-\dfrac{2(g_1-g_2)}{2(f_1-f_2)}=-\dfrac{g_1-g_2}{f_1-f_2}$,

m_2=slope of the line passing thro' the centres $(-g_1, -f_1)$ and $(-g_2, -f_2)$

$$=\frac{(-f_2)-(-f_1)}{(-g_2)-(-g_1)} \qquad \left[\frac{y_2-y_1}{x_2-x_1}\right]$$

*[*P.U. 1955S*]

$$= \frac{f_1-f_2}{g_1-g_2},$$

$$\therefore\ m_1m_2 = -\frac{g_1-g_2}{f_1-f_2}\cdot\frac{f_1-f_2}{g_1-g_2} = -1$$

$\therefore$ the radical axis is $\perp$ to the line joining the centres.

EXAMPLES

1. Show that the radical axis of two circles is a straight line perpendicular to the line joining their centres. [*J. & K. U. 1954*]

2. Find the equation of the radical axis of the two circles $x^2+y^2+2x+4y-7=0$ and $x^2+y^2-6x+2y-5=0$ and show that it is at right angles to the line joining the centres of the two circles. [*P. U.*]

3. Find the length of the common chord of the circles $x^2+y^2-5x-7y+2=0$ and $x^2+y^2-7x-3y+2=0$. Verify that the common chord is at right angles to the line joining the centres of the circles. [*P. U. 1946S*]

89. The radical axes of three circles, taken in pairs, meet in a point.*

Proof. Let the equations of the circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad ...(1)$$
$$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad ...(2)$$
$$x^2+y^2+2g_3x+2f_3y+c_3=0 \quad ...(3)$$

The radical axis of the circles (1) and (2) is

$$2x(g_1-g_2)+2y(f_1-f_2)+(c_1-c_2)=0 \quad ...(4)$$

The radical axis of (2) and (3) is

$$2x(g_2-g_3)+2y(f_2-f_3)+(c_2-c_3)=0 \quad ...(5)$$

And the radical axis of (3) and (1) is

$$2x(g_3-g_1)+2y(f_3-f_1)+(c_3-c_1)=0 \quad ...(6)$$

Adding the L.H.S.'s of (4), (5), (6) vertically, we get

$$2x(0)+2y(0)+0=0$$

$\therefore$ the radical axes meet in a pt. [Art. 42, Cor.]

Radical centre. Def. *The point of concurrence of the radical axes of three circles, taken in pairs, is called the* **radical centre** *of the three circles.*

Example. Find the radical centre of the three circles $x^2+y^2+4x+7=0$, $2x^2+2y^2+3x+5y+9=0$, $x^2+y^2+y=0$. Find the length of the tangent from the radical centre to the second circle. [*P. U. 1941*]

(*i*) The equations of the circles are

$$x^2+y^2+4x+7=0 \quad ...(1)$$

$$2x^2+2y^2+3x+5y+9=0 \text{ or}$$

$$x^2+y^2+\frac{3}{2}x+\frac{5}{2}y+\frac{9}{2}=0 \quad ...(2)$$

[Coeffs. of x^2, $y^2=1$ (*Note this step*)]

$$x^2+y^2+y=0 \quad .. (3)$$

The radical axis of the circles (1) and (2) is [Subtracting (2) from (1)]

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*[*J. & K. U. 1955*]

$\frac{5}{2}x-\frac{5}{2}y+\frac{5}{2}=0$ or $x-y+1=0$ ... (4)

The radical axis of (2) and (3) is [Subtracting (3) from (2)]

$\frac{3}{2}x+\frac{3}{2}y+\frac{9}{2}=0$ or $x+y+3=0$ ...(5)

[The radical centre is the pt. of intersection of (4) and (5).]

Adding (4) and (5), $2x+4=0$, $\therefore \quad x=-2$.

Substituting this value of x in (4), $-2-y+1=0$, $\therefore \quad y=-1$.

$\therefore$ the radical centre is $(-2, -1)$.

[**Check.** The lengths of the tangents from the radical centre $(-2, -1)$ to the circles (1), (2), (3) are equal; thus $\sqrt{4+1-8+7}=\sqrt{4}=2$,

$\sqrt{4+1-3-\frac{5}{2}+\frac{9}{2}}=\sqrt{4}=2$, $\sqrt{4+1-1}=\sqrt{4}=2$.]

(*ii*) The length of the tangent from $(-2, -1)$ to the second circle (2)

$$=\sqrt{(-2)^2+(-1)^2+\frac{3}{2}(-2)+\frac{5}{2}(-1)+\frac{9}{2}}$$

$$=\sqrt{4+1-3-\frac{5}{2}+\frac{9}{2}}=\sqrt{4}=2.$$

EXAMPLES

1. Find the radical centre of the following circles :

(*i*) $x^2+y^2+11x+5y+7=0$, $x^2+y^2+13x+6y+2=0$ and $x^2+y^2+17x+3y-3=0$. [*D. U. 1938*]

(*ii*) $x^2+y^2-4x+7=0$, $2x^2+2y^2-3x+5y+9=0$, and $x^2+y^2+y=0$. [*P. U. 1949S*]

(*iii*) $(x-1)^2+(y-2)^2=1$, $(x+2)^2+(y+1)^2=4$ and $(x-3)^2+(y+4)^2=9$.

****2.** Find the radical centre of the three circles $x^2+y^2-1=0$, $x^2+y^2-4x+6y+2=0$, $x^2+y^2-8x+12y+1=0$. [*P. U. 1946*]

[**Hint.** It will be found that the radical axis of the first and second circles is

$$4x-6y-3=0 \quad ...(1)$$

and the radical axis of the second and third circles is

$$4x-6y+1=0 \quad ...(2)$$

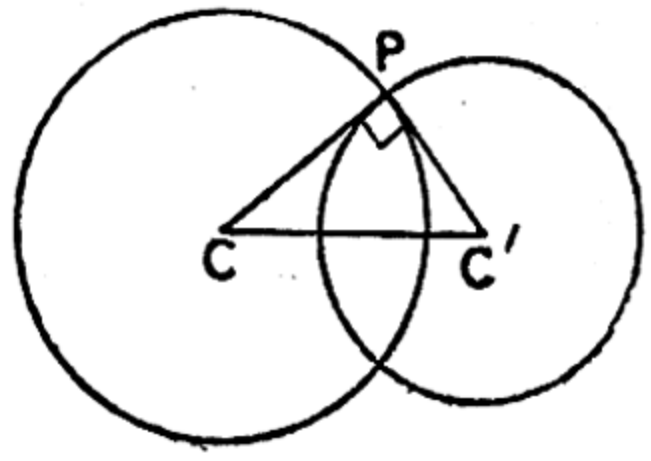
Solving (1) and (2) by cross-multiplication,

$$\frac{x}{-24}=\frac{y}{-16}=\frac{1}{0}, \quad \therefore \; x=\frac{-24}{0}=\infty, \; y=\frac{-16}{0}=\infty.]$$

Orthogonal circles

90. Orthogonal circles. Def. *Two circles are said to be* **orthogonal** *or to cut* **orthogonally** *if the tangents at a point of intersection are at right angles.*

An important fact. If two circles cut orthogonally, the square of the distance between their centres is equal to the sum of the squares of their radii.



Proof. Let C, C′ be the centres of the two circles cutting orthogonally at P so that the tangents to the circles at P are ⊥ [Def. (Art. 90)].

Then the radii CP, C′P, being ⊥ to the tangents at P, are also ⊥,

i.e., $\angle CPC' = 1$ rt. $\angle$

$\therefore$ $CC'^2 = CP^2 + C'P^2$.

Note. The Converse is also seen to be true by reasoning backwards.

91. Condition of orthogonality. To find the condition that the two circles $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$ may cut orthogonally.*

The equations of the circles are

$x^2+y^2+2g_1x+2f_1y+c_1=0$, $x^2+y^2+2g_2x+2f_2y+c_2=0$.

$\therefore$ the centres of the circles are $(-g_1, -f_1)$, $(-g_2, -f_2)$, and their radii are $\sqrt{g_1^2+f_1^2-c_1}$, $\sqrt{g_2^2+f_2^2-c_2}$.

If the circles cut orthogonally, the square of the distance between their centres = the sum of the squares of their radii [Art. 90].

$\therefore$ $(-g_2+g_1)^2+(-f_2+f_1)^2=(g_1^2+f_1^2-c_1)+(g_2^2+f_2^2-c_2)$

or $g_1^2-2g_1g_2+g_2^2+f_1^2-2f_1f_2+f_2^2=g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2$

or $-2g_1g_2-2f_1f_2=-c_1-c_2$

[Cancelling common terms on both sides]

or $2g_1g_2+2f_1f_2=c_1+c_2$,

which is the required condition.

Example. Show that the circles $x^2+y^2-4x-6y+4=0$ and $x^2+y^2-10x-14y+58=0$ intersect at right angles.

Method I. [Method of centre and radius.]

The equations of the circles are $x^2+y^2-4x-6y+4=0$ ... (1)

$x^2+y^2-10x-14y+58=0$ ... (2)

Here $g_1=\frac{1}{2}(-4)=-2$	[$\frac{1}{2}$ coeff. of x]	$g_2=\frac{1}{2}(-10)=-5$,
$f_1=\frac{1}{2}(-6)=-3$,	[$\frac{1}{2}$ coeff. of y]	$f_2=\frac{1}{2}(-14)=-7$,
$c_1=4$.	[Constant term]	$c_2=58$.

$\therefore$ the centres of the circles are (2, 3), (5, 7),

and their radii are $\sqrt{(-2)^2+(-3)^2-4}=\sqrt{4+9-4}=\sqrt{9}=3$,

$\sqrt{(-5)^2+(-7)^2-58}=\sqrt{25+49-58}=\sqrt{16}=4$,

$\therefore$ square of the distance between the centres

$=(5-2)^2+(7-3)^2=(3)^2+(4)^2=9+16=25$,

also sum of the squares of the radii $=(3)^2+(4)^2=9+16=25$

$\therefore$ the circles cut orthogonally [Art. 90].

Method II. [Method of condition of orthogonality.]

The equations of the circles are $x^2+y^2-4x-6y+4=0$ ... (1)

$x^2+y^2-10x-14y+58=0$ ... (2)

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* [*P. U. 1951S*]

Here $g_1=\frac{1}{2}(-4)$ [$\frac{1}{2}$ coeff. of x] $=-2$, | $g_2=\frac{1}{2}(-10)=-5$,
$f_1=\frac{1}{2}(-6)$ [$\frac{1}{2}$ coeff. of y] $=-3$, | $f_2=\frac{1}{2}(-14)=-7$,
$c_1=4$. [Constant term] | $c_2=58$.

Substituting these values in the condition of orthogonality $2g_1g_2+2f_1f_2=c_1+c_2$ (Art. 91), we get

$2(-2)(-5)+2(-3)(-7)=4+58$ or $20+42=4+58$

or $62=62$, which is true

$\therefore$ the circles cut orthogonally.

EXAMPLES

1. Determine the condition that the circles

$$\lambda x^2+\lambda y^2+2ax+2by+c=0$$

and $$x^2+y^2+2a'x+2b'y+c'=0$$

may intersect orthogonally. [*P.U. 1954S*]

2. Show that the circles

$$\mathbf{x^2+y^2-4x-6y-12=0 \text{ and } x^2+y^2+6x+4y-12=0}$$

are orthogonal to each other. [*P. U. 1945*]

3. *Find the equation of the circle which passes through the origin and cuts orthogonally each of the circles*

$$x^2+y^2-8y+12=0 \text{ and } x^2+y^2-4x-6y-3=0.$$ [*M.U. 1933*]

4. Find the equation of the circle through the points (2, 0), (0, 2) and orthogonal to the circle $2x^2+2y^2+5x-6y+4=0$. [*P.U. 1957S*]

Systems of circles

92. Any circle through the intersections of two given circles. *To find the equation of any circle through the points of intersection of the two circles*

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

and

$$x^2+y^2+2g_2x+2f_2y+c_2=0.$$

[**K-Method.**]

The equations of the circles are

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad \ldots (1)$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad \ldots (2)$$

Consider the equation

$$x^2+y^2+2g_1x+2f_1y+c_1+k(x^2+y^2+2g_2x+2f_2y+c_2)=0 \quad \ldots (3)$$

(*i*) It is an equation of the second degree in x and y, the coefficients of x^2 and y^2 are equal (each being $=1+k$), and there is no term involving the product xy, $\therefore$ it represents a circle [Art. 58].

(*ii*) The co-ordinates of the pts. which satisfy both (1) and (2) also satisfy (3) [$\because$ substituting from (1) and (2) in (3), we get $0+k(0)=0$, which is true]

$\therefore$ the pts. of intersection of the circles (1) and (2) lie on (3).

$\therefore$ (3) is the equation of a circle passing thro' the pts. of intersection of the circles (1) and (2).

[**Aid to memory.** *The equation of any circle thro' the pts. of intersection of two given circles is* **one circle+k (other circle)=0,**
where "one circle" stands for "the L.H.S. of the equation of one circle (R.H.S. being zero)" and so for the "other circle". (*Cf.* Art. 43, Aid to memory.)]

Note. The value of k is found from the second condition satisfied by the circle.

EXAMPLES

1. *Find the equation of the circle through the points of intersection of the circles* $x^2+y^2+2x+3y-7=0$ *and* $x^2+y^2+3x-2y-1=0$ *and through the point* $(1, 2)$. [*Pesh. U. 1954*]

[**Sol.** The equations of the circles are

$$x^2+y^2+2x+3y-7=0 \quad \ldots (1)$$
$$x^2+y^2+3x-2y-1=0 \quad \ldots (2)$$

(*i*) The equation of *any* circle thro' the pts. of intersection of (1) and (2) is $x^2+y^2+2x+3y-7+k(x^2+y^2+3x-2y-1)=0 \quad \ldots (3)$

[one circle$+k$(other circle)$=0$]

(*ii*) If it passes thro' (1, 2), then

$1+4+2+6-7+k(1+4+3-4-1)=0$ or $6+3k=0$, $\therefore \quad k=-2$.

(*iii*) Substituting this value of k in (3),

$$x^2+y^2+2x+3y-7-2(x^2+y^2+3x-2y-1)=0$$

or $-x^2-y^2-4x+7y-5=0$ or $x^2+y^2+4x-7y+5=0$,

which is the required equation.]

2. *Find the equation of the circle through the points of intersection of the circles* $x^2+y^2-6x=0$ *and* $x^2+y^2+4y-1=0$ *and*

(*i*) *passing through the point* $(-1, 1)$; [*M. U.*]

(*ii*) *having its centre on the line* $2x+3y=0$.

3. Find the equation of a circle through the points of intersection of $x^2+y^2-1=0$ and $x^2+y^2-2x-4y+1=0$ and touching the line $x+2y=0$.

93. Any circle through the intersections of a given circle and a given line.

To find the equation of any circle through the points of intersection of the circle $x^2+y^2+2gx+2fy+c=0$ *and the line* $Ax+By+C=0$.

[K-Method.]

The equations of the circle and the line are

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$
$$Ax+By+C=0 \quad \ldots(2)$$

Consider the equation

$$x^2+y^2+2gx+2fy+c+k(Ax+By+C)=0 \quad \ldots(3)$$

(*i*) It is an equation of the second degree in x and y, the coefficients of x^2 and y^2 are equal (each being $=1$), and there is no term involving the product xy. $\therefore$ it represents a circle [Art. 58].

(*ii*) The co-ordinates of the pts. which satisfy both (1) and (2) also satisfy (3) [$\because$ substituting from (1) and (2) in (3), we get $0+k(0)=0$, which is true]

$\therefore$ the pts. of intersection of the circle (1) and the line (2) lie on (3).

$\therefore$ (3) is the equation of a circle passing thro' the pts. of intersection of the circle (1) and the line (2).

[Aid to memory. *The equation of any circle thro' the pts. of intersection of a given circle and a given line is* **circle+k (line)=0,**
where "circle" stands for "the L.H.S. of the equation of the circle (R.H S. being zero)" and so for the "line." (*Cf.* Arts. 43, 92, Aid to memory.)]

Note. The value of k is determined from the second condition satisfied by the circle.

EXAMPLES

1. Find the equation of a circle passing through the points of intersection of $2x+y-10=0$ and $x^2+y^2=25$ and the origin. [*P.U. 1939*]

2. Find the equation of the circle which passes through $(2a, 0)$ and whose radical axis in relation to $x^2+y^2-a^2=0$ is $x=\frac{a}{2}$. [*A. U. 1942*]

****3. The equation of the chord of the circle $x^2+y^2=a^2$ is $x\cos\alpha+y\sin\alpha=p$, find the equation of the circle described on this chord as diameter.** [*A. U. 1940*]

[**Hint.** The equation of *any* circle thro' the pts. of intersection of the given circle and the given chord is

$$x^2+y^2-a^2+k(x\cos\alpha+y\sin\alpha-p)=0 \quad \ldots(1)$$

If its centre lies on $x\cos\alpha+y\sin\alpha=p$, $k=?$ Substitute this value of k in (1).]

MISCELLANEOUS EXAMPLES ON CHAPTER V

1. *Find the equation of the circle which touches each axis at a distance 2 from the origin.*

[**Explanation.** "At a distance 2" means "at a distance $+2$".]

2. Find the equation of the circle inscribed in the triangle the equations of whose sides are $x=1$, $y=2$ and $3x-4y=7$.

3. Find the co-ordinates of the centre and the radius of the circle :

(*i*) $3(x^2+y^2)-6x+8y-5=0$. [*P. U. 1946S*]

(*ii*) $\sqrt{1+a^2}\,(x^2+y^2)-2bx-2aby=0$.

4. Show that the locus of a point the sum of the squares of whose distances from three fixed points is constant is a circle. Find the co-ordinates of the centre of the circle. [*C. U. 1952*]

5. A point moves so that the sum of the squares of its distances from n fixed points is given. Prove that its locus is a circle.

[**Note. Important.** *For problems relating to n fixed points, let the fixed points be* (x_1, y_1), (x_2, y_2), ..., (x_n, y_n).]

6. *A point moves so that the sum of the squares of its distances from the four sides of a square is constant. Show that the locus of the point is a circle.*

[**Hint.** See Art. 8-I, Note. Take the centre of the square as origin and the lines thro' it $\parallel$ to the sides as axes. Let each side $=2a$. Then the equations of the sides are $y=a$, $y=-a$; $x=a$, $x=-a$ (Arts. 15, 16).]

7. Whatever be the value of α, prove that the locus of the intersection of the straight lines $x\cos\alpha+y\sin\alpha=a$ and $x\sin\alpha-y\cos\alpha=b$ is a circle. [*C. U. 1953*]

8. Circumcircle. Find the equation of the circle circumscribing the triangle formed by the lines $3x+y-5=0$, $x+y+1=0$ and $2x+y-4=0$. [*D. U. 1938*]

9. Find the equations of the tangents to $x^2+y^2=25$ at the points (3, 4) and (4, −3) ; and prove that they intersect at right angles at the point (7, 1).

10. *Find the equation of the circle whose centre is at the point* (a, b) *and which passes through the origin. Write down the equations of the tangent and normal at the origin.* [*A. B. 1951*]

11. Find the equation of the circle which touches the x-axis at the point (3, 0) and passes through another point (1, 2). Also verify that the normal at the given point to the above circle passes through its centre. [*P. U. 1947*]

[**Hint.** Let the required equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$

It meets the x-axis where, putting $y=0$,

$$x^2+2gx+c=0 \quad \ldots(2)$$

But the circle *touches* the x-axis at (3, 0), *i.e.*, where $x=3$, $\therefore$ (2) is the same as $(x-3)^2=0$ or $x^2-6x+9=0$ ...(3)

$\therefore$ comparing coefficients in (2) and (3),

$$2g=-6,\ c=9 \quad \ldots(4)$$

Again $\because$ the circle passes thro' (1, 2),

$\therefore$ from (1), $1+4+2g+4f+c=0$ ...(5)

Find the values of g, f, c from (4) and (5), and substitute them in (1).]

12. *Find the equation of the circle which passes through the origin and the point* (2, 1) *and touches the straight line* $x=y$ *at the origin.* [*P. U. 1944*]

13. Intercepts of a circle on the axes. Find the intercepts made on the axes by the circle $x^2+y^2+2gx+2fy+c=0$.

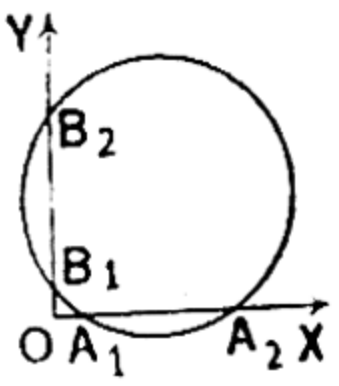
[**Sol.** The equation of the circle is

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots(1)$$

It meets the x-axis where, putting $y=0$ in (1),

$$x^2+2gx+c=0,$$

which is a quadratic in x. Let x_1, x_2 be the roots.



Then $x_1+x_2=-2g,\ x_1x_2=c,$

$\therefore\ A_1A_2=OA_2-OA_1=x_2-x_1$

$$=\sqrt{(x_2+x_1)^2-4x_2x_1}=\sqrt{4g^2-4c}=2\sqrt{g^2-c},$$

which is the intercept on the x-axis.

Again, the circle meets the y-axis where, putting $x=0$ in (1),

$y^2+2fy+c=0$, which is a quadratic in y. Let y_1, y_2 be the roots.

Then $y_1+y_2=-2f,\ y_1y_2=c,$

$\therefore\ B_1B_2=OB_2-OB_1=y_2-y_1=\sqrt{(y_2+y_1)^2-4y_2y_1}=\sqrt{4f^2-4c}$

$$=2\sqrt{f^2-c},$$

which is the intercept on the y-axis.]

14. Find the equation to the circle which has as diameter the join of the points (0, 1) and (2, −3). Find the lengths of the chords that it cuts off from the axes.

15. Prove that the straight line $y=x+a\sqrt{2}$ touches the circle $x^2+y^2=a^2$, and find its point of contact. [*B.H.U. 1951*]

16. Show that the circle $x^2+y^2-2ax-2ay+a^2=0$ touches the axes of x and y. [*A.U. 1944*]

17. Prove that the tangent to the circle $x^2+y^2=5$ at the point $(1, -2)$ also touches the circle $x^2+y^2-8x+6y+20=0$ and find the co-ordinates of the point of contact. [*Pesh. U. 1952*]

18. Find the condition that the straight line $cx-by+b^2=0$ may touch the circle $x^2+y^2=ax+by$ and find the point of contact.

[**Hint.** Use the method of intersections. It will be found that $x^2\left(1+\frac{c^2}{b^2}\right)+x(c-a)=0$ ...(1) has equal roots. $\therefore$ $(c-a)^2-4(0)=0$, $\therefore$ $c=a$, which is the required condition.

Substituting this value of c in (1), $x^2=0$, which has equal roots, each$=0$, $\therefore$ the line touches the circle at the pt. where $x=0$. Substituting this value of x in the equation of the line, $y=b$.]

19. Find the equation to the circle which touches each axis and is of radius a. [*P.U. 1946*]

[**Hint.** Let the required equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad ...(1)$$

It meets the x-axis where, putting $y=0$ in (1),

$$x^2+2gx+c=0 \quad ...(2)$$

$\because$ the circle touches the x-axis, (2) has equal roots.

$\therefore$ $g^2=c$ ...(3). Similarly $f^2=c$ ...(4). Also $\sqrt{g^2+f^2-c}=a$ ...(5)

From (4) and (5), $\sqrt{g^2}=a$ or, squaring, $g^2=a^2$. (*Note this step*)

$\therefore$ $g=\pm a$. From (3) and (5), $\sqrt{f^2}=a$ or, squaring, $f^2=a^2$,

$\therefore$ $f=\pm a$. From (3), $a^2=c$. Substitute these values of g, f, c in (1).

****Or thus** : Draw a rough Fig. Join the centre of the circle to its pts. of contact with the axes. The centre of the circle is (a, a) (first quadrant), $(-a, a)$ (second quadrant), $(-a, -a)$ (third quadrant), $(a, -a)$ (fourth quadrant), and the radius$=a$.]

20. Find the equation of the circle which touches the lines

(*i*) **x=0, y=0, and x=c.** [*D.U. 1946*]

(*ii*) **x=0, x=a and 3x+4y+5a=0.** [*D.U. 1947*]

[**Hint.** (*i*) Let the required equation of the circle be

$$x^2+y^2+2gx+2fy+k^*=0.$$

Proceed as in Ex. 19.

(*ii*) Proceed as in Ex. 19. It will be found that $f^2=c$ ...(1)

$f^2=a^2+2ga+c$, $\therefore$ from (1), $0=a^2+2ga$ or $g=-\frac{a}{2}$ ...(2). Again, by the tangent property, $\pm\frac{-3g-4f+5a}{5}=\sqrt{g^2+f^2-c}=g$ [From (1)]

or $\pm\frac{\frac{3a}{2}-4f+5a}{5}=-\frac{a}{2}$. (From (2))]

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*Why to use k (and not c). In order to avoid confusion with the given c in $x=c$, we take the equation of the circle as $x^2+y^2+2gx+2fy+k=0$ (and *not* $x^2+y^2+2gx+2fy+c=0$).

21. (*i*) **Show that the line $y=m(x-a)+a\sqrt{1+m^2}$ touches the circle $x^2+y^2=2ax$, whatever the value of m may be.**

(*ii*) **Find whether the straight line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y+1=0$.**

[**Hint.** (*i*) Show that the length of the $\perp$ from the centre of the circle on the line, *in magnitude*=the radius.]

22. Three tangents to the circle $x^2+y^2=25$ form an equilateral triangle and one of them is parallel to the x-axis. Find their equations.

23. Find the equation of the tangent to the circle $x^2+y^2=a^2$ which

(*i*) **is parallel to $y=mx+c$;**

(*ii*) **is perpendicular to $y=mx+c$;** [*P (P). U. 1953*]

(*iii*) **passes through the point (b, 0).**

24. Show that $7y-x=5$ touches the circle $x^2+y^2-5x+5y=0$ and find the equation of the parallel tangent. [*P.U. 1950S*]

25. Find the equations of the two tangents from $(-5, 0)$ to the circle whose radius is 3 and whose centre is at the origin. [*P.U. 1941*]

26. Find the equations of the tangents from the origin to the circle $x^2+y^2+10(x+y)+40=0$ and the angle between them. [*B.U.*]

27. Find the co-ordinates of the points where the line $x+2y=5$ meets the circle $x^2+y^2=50$; find the equations of the tangents at these points and show that they intersect in the point (10, 20). Write down the equation of the polar of the point (10, 20) with respect to the circle.

28. If the polar of the point (x', y') with respect to the circle $x^2+y^2=a^2$, touches the circle $(x-a)^2+y^2=a^2$, show that (x', y') lies on the curve $y^2+2ax=a^2$. [*A.U. 1938*]

29. The tangents at P and Q lying on the circle $x^2+y^2=9$ intersect at (5, 4). Show that PQ cuts the x-axis at $\left(\frac{9}{5}, 0\right)$. [*A.U. 1942*]

30. Find the pole of

(*i*) **the line $x-2y=3$ with respect to the circle $4x^2+4y^2=5$.**

(*ii*) **the line $\frac{x}{a}+\frac{y}{b}=1$ with respect to the circle $x^2+y^2=c^2$.**

[*A. U. 1943*]

31. Find the co-ordinates of the point of intersection of the tangents drawn to the circle $x^2+y^2=a^2$ at the points where it is cut by the straight line $x\cos\alpha+y\sin\alpha=p$. [*A. U.*]

32. Equation of the common chord. *Find the equation of the common chord of the circles $x^2+y^2-3x-y+2=0$ and $x^2+y^2-2x+y+3=0$.* [*D. U.*]

33. Touching circles. **Show that the circles $x^2+y^2-6x+5=0$ and $x^2+y^2-8y+7=0$ touch. Find the equation of the tangent at the point of contact and the co-ordinates of the point of contact.** [*B. U.*]

34. (*i*) **Prove that the circles**

$$(x-a)^2+(y-b)^2=c^2 \text{ and } (x-b)^2+(y-a)^2=c^2$$

touch each other, if $(a-b)^2=2c^2$.

(*ii*) **Show that the two circles $x^2+y^2+4x+2y=15$ and $x^2+y^2=5$ touch internally ; and find their point of contact.** [*M. B. 1952*]

35. Show that the following pairs of circles intersect orthogonally :

(*i*) $x^2+y^2+2ax+c=0$ and $x^2+y^2+2by-c=0$;

(*ii*) $x^2+y^2-2ax+2by+c=0$ and $x^2+y^2+2bx+2ay-c=0$.

****[Angle of intersection of two circles. Def.* *The* angle of intersection of two circles *is the angle between the tangents to the circles at a point of intersection.*]**

****36. Angle of intersection formula for two circles.** *If r, r' are the radii of two circles, d the distance between the centres and θ the angle of intersection, then*

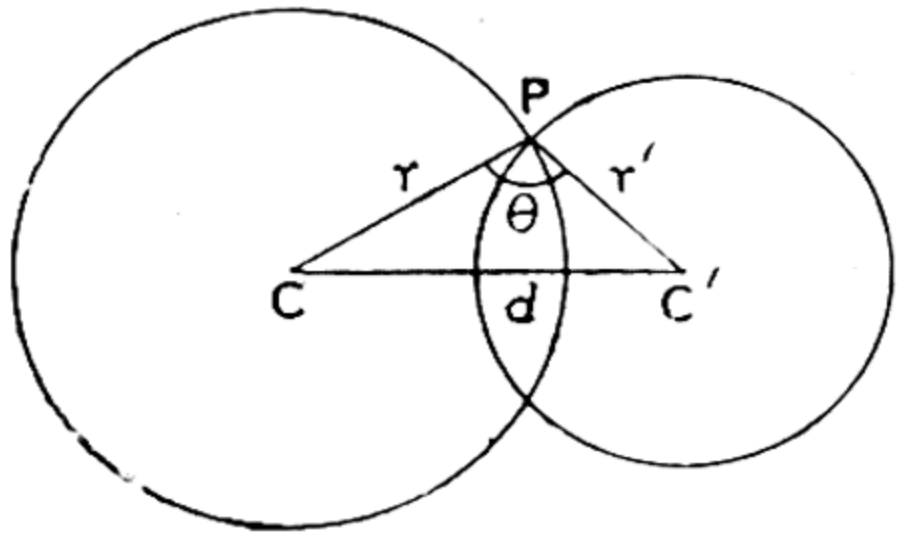
$$\cos\theta=\frac{r^2+r'^2-d^2}{2rr'}.$$

[**Sol.** Let C, C′ be the centres of the circles and P a pt. of intersection.

Then $CP=r$, $C'P=r'$, $CC'=d$.

The tangents at P being $\perp$ to the radii thro' P, the angle between the tangents=the angle between the radii†.

$\therefore \angle CPC'=\theta$.



In the $\triangle CPC'$, by the cosine formula,

$$\cos\theta=\frac{CP^2+C'P^2-CC'^2}{2CP.C'P}$$

$$[\cos A=\frac{b^2+c^2-a^2}{2bc} \text{ (Intermediate Trigonometry)}]$$

$$=\frac{r^2+r'^2-d^2}{2rr'}.]$$

****37. Angle of intersection of two circles.** Find the angle at which the circles $x^2+y^2-2x-4y=4$ and $x^2+y^2=16$ intersect. [*P. U. 1953*]

[**Hint.** It will be found that the centres of the circles are (1, 2), (0, 0), and their radii are 3, 4.

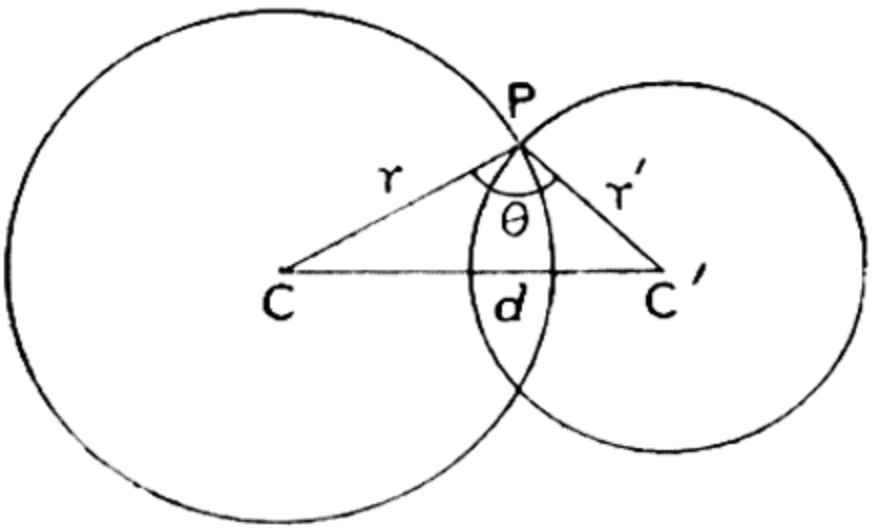
$\therefore r=3$, $r'=4$,

d, the distance between the centres,

$$\begin{matrix}(1)&(2)\\(1,\ 2)&(0,\ 0)\end{matrix}$$

$$=\sqrt{(0-1)^2+(0-2)^2}$$

$$=\sqrt{5}.$$



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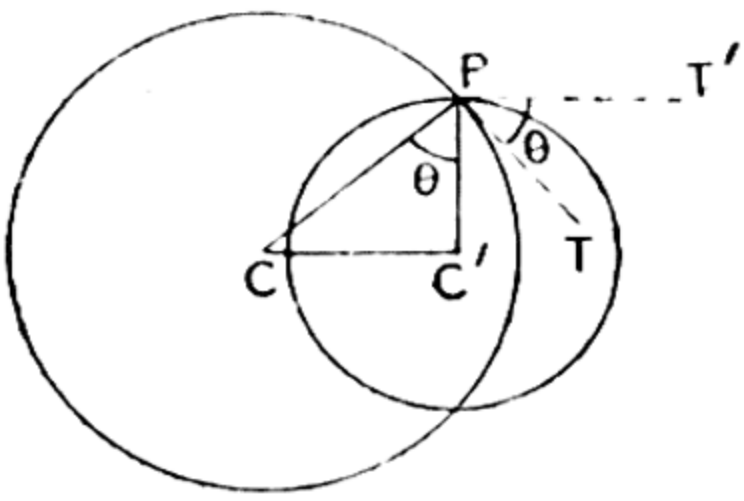
*[*P. U. 1954S*]

†For in the adjoining Fig.,

$\angle CPT=\angle C'PT'$ ($\because$ each $=90°$)

$\therefore$ taking away the common $\angle C'PT$,

$\angle CPC'=\angle TPT'$.



Now $\cos\theta=\dfrac{r^2+r'^2-d^2}{2rr'}=?$]

38. **Find the equation of the circle described on the common chord of the circles $x^2+y^2-4x-6y-12=0$ and $x^2+y^2+6x+4y-12=0$ as a diameter. [*P. U. 1945*]

[**Hint.** (*i*) The equation of *any* circle thro' the pts. of intersection of the given circles is $x^2+y^2-4x-6y-12+k(x^2+y^2+6x+4y-12)=0$...(1)

(*ii*) If its centre $\left(-\dfrac{-2+3k}{1+k}, -\dfrac{-3+2k}{1+k}\right)$ lies on the common chord, *i.e.*, radical axis $-10x-10y=0$ or $x+y=0$, find the value of k.

(*iii*) Substitute this value of k in (1).

Or thus : **Short-cut. Find the equation of the radical axis of the two circles.

(*i*) Write down the equation of *any* circle thro' the pts. of intersection of the first circle and the radical axis, and proceed as in steps (*ii*) and (*iii*).]

CHAPTER VI
THE PARABOLA

SECTION I
EQUATION OF A PARABOLA

****94.** (*a*) **Conic. Geometrical Def.** A *conic section* or **conic** is the locus of a point which moves* so that its distance from a fixed point is in a constant ratio to its distance from a fixed straight line.

[Compare the *analytical definition* of a conic (Art. 63).]

Defs. and notation. (*i*) The fixed point is called the **focus,** and is usually denoted by S.

(*ii*) The fixed straight line is called the **directrix.**

(*iii*) The constant ratio is called the **eccentricity,** and is denoted by *e* (the initial letter of the word *eccentricity*).

***Note 1.** It is assumed that the point moves in the plane in which the fixed point and the fixed straight line lie.

****Note 2. Why the name "conic section".** Since the *section* of a right circular *cone*, which is cut by a plane, is a conic as defined above, hence the name **"conic section."**†

****(*b*) Three particular cases of a conic.**

Case I. If the eccentricity $e=1$, the conic is called a **parabola.**

Case II. If $e < 1$, it is called an **ellipse.**

Case III. If $e > 1$, it is called a **hyperbola.** Hence we have

(*c*) **Parabola. Def.**§ *A* **parabola** *is the locus of a point which moves*‡ *so that its distance from a fixed point is equal to its distance from a fixed straight line.*

Defs. (*i*) The fixed point is called the **focus,** and is usually denoted by S.

(*ii*) The fixed straight line is called the **directrix.**

‡**Note.** It is assumed that the point moves in the plane in which the fixed point and the fixed straight line lie.

[**N.B.** *If P is any point on a parabola whose focus is S, and PM* ⊥ *on the directrix, then* **SP=PM.**.]

Axis. Def. *The* **axis** *of a parabola* (*or conic*) *is the straight line through the focus perpendicular to the directrix.*

Vertex. Def. *The* **vertex** *of a parabola* (*or conic*) *is the point of intersection of the parabola* (*or conic*) *with the axis.*

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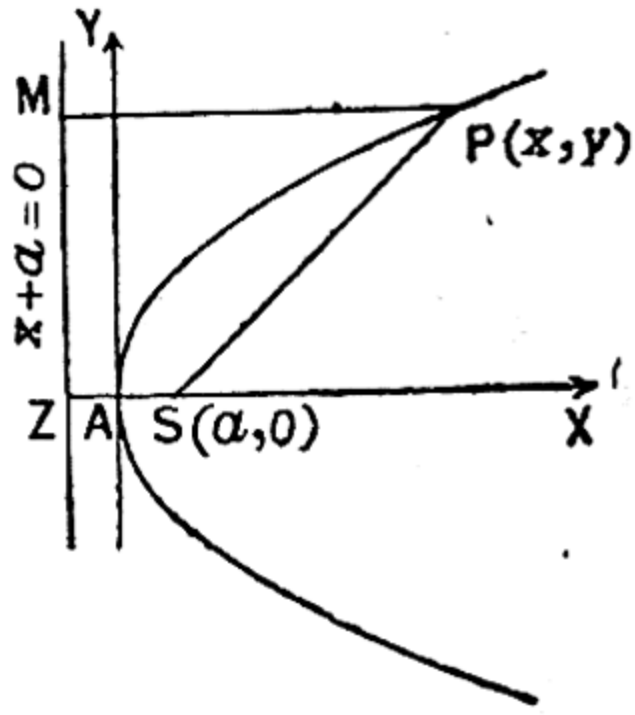
†The first systematic treatment was given by Euclid (323—284 B. C.), the "father" of *Demonstrative Geometry,* in a book now lost.

§[*P*(*P*). *U. 1955*]

95-I. Standard form. To find the equation of a parabola in the form $y^2=4ax$.*

Let S be the focus and ZM the directrix of the parabola. From S draw SZ $\perp$ on ZM, and bisect SZ in A.

[Then A lies on the parabola (Def.) $\because$ SA=AZ]



Let SZ=2*a*, then AS=AZ=*a*.

(*i*) Take A as origin†, AS as the *x*-axis, and AY $\perp$ to it as the *y*-axis.

Then the co-ordinates of S are (*a*, 0), and the equation of ZM is $x=-a$
[(Art. 16) $\parallel$ to *y*-axis at a distance *a* to the *left*]
or $x+a=0$.

(*ii*) Let P (x, y) be *any* pt. on the parabola. Join SP, and from P draw PM $\perp$ on ZM. Then SP=PM ...(1) [Def. (Art. 94, (*c*))]

But $SP=\sqrt{(x-a)^2+(y-0)^2}=\sqrt{(x-a)^2+y^2}$,

and $PM=\dfrac{x+a}{1}$ [$\perp$ from (x, y) on $x+a=0$]

$=x+a$.

$\therefore$ from (1), $\sqrt{(x-a)^2+y^2}=x+a$

or, squaring, $x^2-2ax+a^2+y^2=x^2+2ax+a^2$

or $\mathbf{y^2=4ax}$, which is the required equation.

Cor. Focal distance. *The focal distance of any point $P(x_1, y_1)$ on the parabola is* $\mathbf{SP=a+x_1}$.*

For, the focal distance of any pt. P(x, y) on the parabola is SP=PM=$x+a$ (proved above).

[**Important.** *The focus lies on the axis and* **AS=a;** *the directrix ZM is $\perp$ to the axis and* **AZ=a** *on the opposite side of A as S.*]

[**N.B.** *The focus of the parabola $y^2=4ax$ is* **(a, 0)**, *the equation of the directrix is* **x=−a**, *the equation of the axis (x-axis) is* **y=0**, *and the vertex (origin) is* **(0, 0)**.]

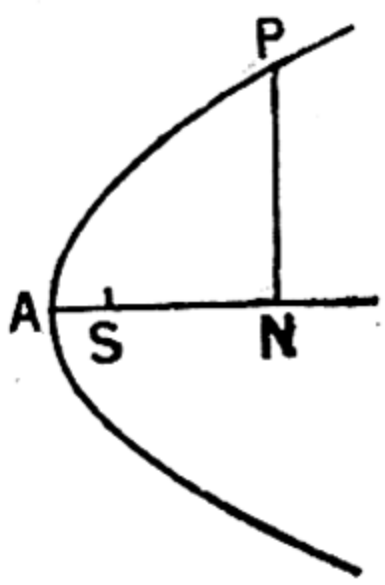
Note 1. Standard form. Since $y^2=4ax$ is the simplest form of the equation of a parabola, it is called the **standard form.**

Note 2. To interpret geometrically the equation $y^2=4ax$.

From *any* pt. P on the parabola draw PN $\perp$ on the axis, then

$\because$ $y^2=4ax$

$\therefore$ $PN^2=4$ AS.AN, which is the required geometrical interpretation.



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*[*J. & K. U. 1957*]

†**Why this step.** To find the co-ordinates of the focus S and the equation of the directrix ZM.

EXAMPLES

1. Find the value of a when the parabola $y^2=4ax$ passes through the point $(1, -2)$.

2. Find the point of the parabola $y^2=8x$, whose ordinate is twice the abscissa.

3. Shew that the area of the triangle inscribed in the parabola $y^2-4ax=0$ is $\frac{1}{8a}(y_1-y_2)(y_2-y_3)(y_3-y_1)$,

where y_1, y_2, y_3 are the ordinates of the angular points. [*A.U. 1946*]

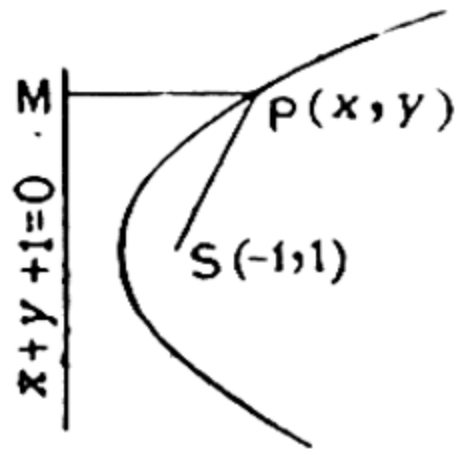
****4.** The focal distance of a point on the parabola $y^2=8x$ is 8 ; find its co-ordinates. [*J. & K. U. 1956*]

95-II. Equation of a parabola referred to any focus and directrix.

Example. Obtain the equation of a parabola whose focus is the point $(-1, 1)$ and whose directrix is the straight line $x+y+1=0$. [*C. U. 1932*]

Let S$(-1, 1)$ be the focus.

(*i*) Let P(x, y) be *any* pt. on the parabola. Join SP, and from P draw PM $\perp$ on the directrix $x+y+1=0$.



(*ii*) Then SP=PM ...(1) [Def. (Art. 94, (*c*))]

(*iii*) But $SP=\sqrt{(x+1)^2+(y-1)^2}$, [Distance formula]

and $PM=\frac{x+y+1}{\sqrt{2}}$ [$\perp$ distance formula]

$\therefore$ from (1), $\sqrt{(x+1)^2+(y-1)^2}=\frac{x+y+1}{\sqrt{2}}$

or, squaring, $x^2+2x+1+y^2-2y+1=\frac{x^2+y^2+1+2xy+2y+2x}{2}$*

or $2(x^2+y^2+2x-2y+2)=x^2+y^2+2xy+2x+2y+1$

or $x^2-2xy+y^2+2x-6y+3=0$, which is the required equation.

EXAMPLES

1. What is the equation of the parabola whose focus is $(-3, 0)$ and the directrix $x+5=0$? [*J. & K. U. 1953*]

2. Find the equation to the parabola with focus $(3, -4)$ and directrix $6x-7y+5=0$. Simplify the result. [*P. U. 1943S*]

96. Tracing a parabola.

To trace the curve $y^2=4ax$.

The equation of the curve is $y^2=4ax$ ...(1)

(*i*) **Axes-intersections.** The x-axis meets the curve where, putting $y=0$ in (1). $x=0$, *i.e.*, in (0, 0) (origin).

Again, the y-axis meets the curve where, putting $x=0$ in (1), $y^2=0$, which has equal roots, each $=0$, $\therefore$ the y-axis *touches* the curve at the pt. where $y=0$, *i.e.*, at (0, 0) (origin).

(*ii*) **Symmetry.** From (1), $y=\pm 2\sqrt{a}\sqrt{x}$ ...(2)

***How to write this step.** $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$.

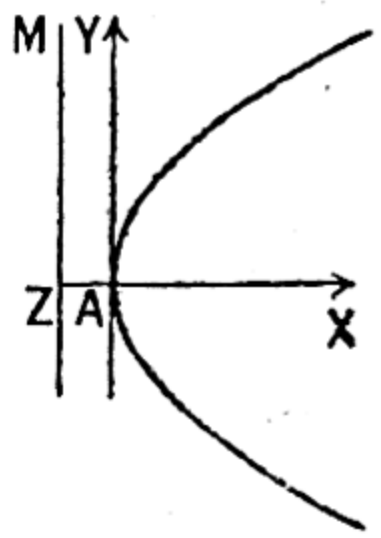
$\therefore$ for every +ve value of x, there are two equal and opposite values of y, $\therefore$ the curve is symmetrical about the x-axis.

(*iii*) **Imaginary values.** From (2), y is imaginary when x is − ve.

$\therefore$ no portion of the curve lies to the left of the y-axis.

(*iv*) **Space occupied.** From (2), taking +ve sign with the square root, $y=2\sqrt{a}\sqrt{x}$. $\therefore$ as x (+ve) increases, y increases ; when $x=\infty$, $y=\infty$, $\therefore$ the curve extends to ∞ to the right of the y-axis, *i.e.*, it is open.

Hence the shape of the curve is that shown in the Fig.



It is a parabola whose axis runs in the +ve direction of the x-axis.

Note. Right-handed parabola. Since the parabola $y^2=4ax$ lies wholly to the *right-hand* side of the y-axis, it is called the **right-handed parabola.**

[Rule to trace a parabola (or conic) whose equation is given :

(*i*) **Axes-intersection.** *The x-axis meets the curve where, putting $y=0$ in the equation of the curve, $x=?$ Again, the y-axis meets the curve where, putting $x=0$ in the equation of the curve, $y=?$*

(*ii*) **Symmetry.** *Solve the equation for y in terms of x. If for every value of x, there are two equal and opposite values of y, the curve is symmetrical about the x-axis. Again, solve the equation for x in terms of y. If for every value of y, there are two equal and opposite values of x, the curve is symmetrical about the y-axis.*

(*iii*) **Imaginary values.** [*See step* (*ii*).] *From the value of y in terms of x, find the values of x for which y is imaginary. Again, from the value of x in terms of y, find the values of y for which x is imaginary.*

(*iv*) **Space occupied.** *If, as x increases, y increases ; when $x=\infty$, $y=\infty$, the curve extends to ∞, i.e., it is open, otherwise it is closed.*]

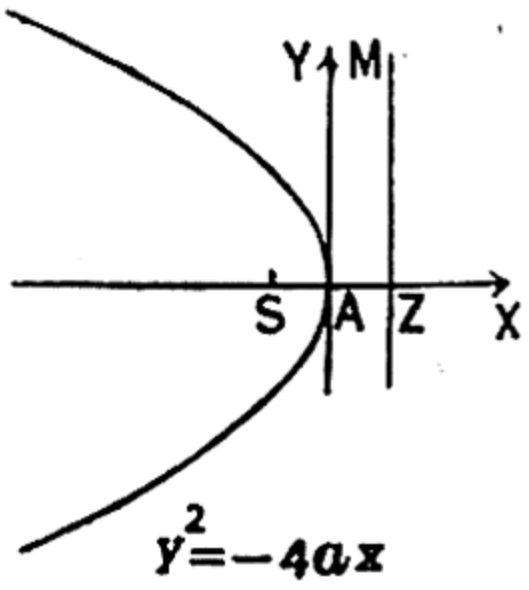
[Aid to memory. Remember the word AS-IS. A is the initial letter of the word *Axes-intersections*, S of *Symmetry*, I of *Imaginary values*, and S of *Space occupied.*]

EXAMPLES

1. Trace the curve $y^2=-4ax$.

[Hint. Proceed as in Art. 96. Here in step (*iii*), *Imaginary values*, $y=\pm 2\sqrt{a}\sqrt{-x}$, $\therefore$ y is imaginary when x is +ve, $\therefore$ no portion of the curve lies to the right of the y-axis.

The shape of the curve is that shown in the Fig.



It is a parabola whose axis runs in the −ve direction of the x-axis.

Note. Left-handed parabola. Since the parabola $y^2=-4ax$ lies wholly to the *left hand* side of the y-axis, it is called the **left-handed parabola.**

Cor. To find the focus and the directrix of the parabola $y^2=-4ax$. The focus lies on the axis, and AS$=a$, $\therefore$ S is $(-a, 0)$. Again, the directrix ZM is $\perp$ to the axis, AZ$=a$ on the oppo-

site side of A as S, ∴ the equation of the directrix is **x=a** (Art. 16).]

2. Trace the curve $x^2=4ay$.

[**Sol.** The equation of the curve is $x^2=4ay$ ...(1)

(*i*) **Axes-intersections.** The x-axis meets the curve where, putting $y=0$ in (1), $x^2=0$, which has equal roots, each $=0$, ∴ the x-axis *touches* the curve at the pt. where $x=0$, *i.e.*, at (0, 0) (origin).

Again, the y-axis meets the curve where, putting $x=0$ in (1), $y=0$, *i.e.*, in (0, 0) (origin).

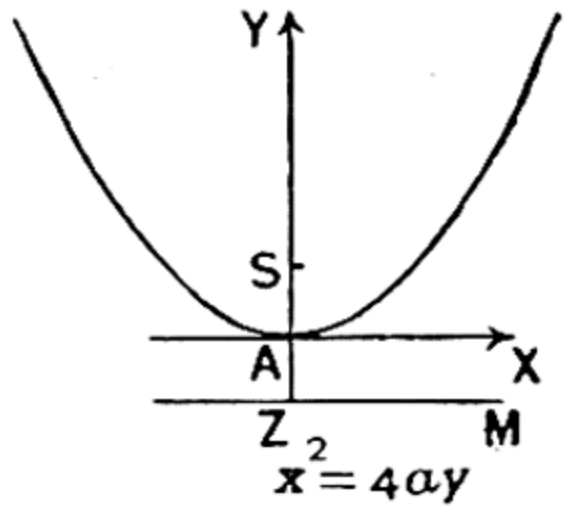
(*ii*) **Symmetry.** From (1), $x=\pm 2\sqrt{a}\sqrt{y}$ ...(2)

∴ for every +ve value of y, there are two equal and opposite values of x, ∴ the curve is symmetrical about the y-axis.

(*iii*) **Imaginary values.** From (2), x is imaginary when y is −ve, ∴ no portion of the curve lies below the x-axis.

(*iv*) **Space occupied.** From (2), taking +ve sign with the square root, $x=2\sqrt{a}\sqrt{y}$, ∴ as y(+ve) increases, x increases ; when $y=\infty$, $x=\infty$, ∴ the curve extends to ∞ above the x-axis, *i.e.*, it is open.

Hence the shape of the curve is that shown in the Fig.



$x^2=4ay$

It is a parabola whose axis runs in the +ve direction of the y-axis.

Note. Upward parabola. Since the parabola $x^2=4ay$ lies wholly *upwards* of the x-axis, it is called the **upward parabola.**

Cor. To find the focus and the directrix of the parabola $x^2=4ay$. The focus lies on the axis, and AS$=a$, ∴ S is **(0, a)**.

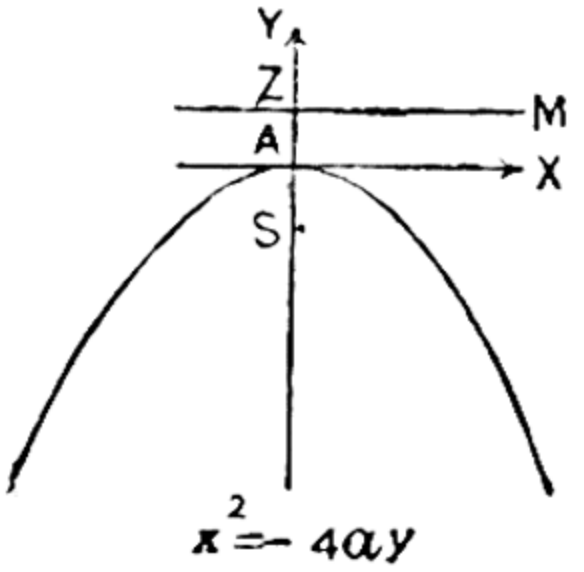
Again, the directrix ZM is ⊥ to the axis, AZ$=a$ on the opposite side of A as S, ∴ the equation of the directrix is **y=−a** (Art. 15).

Note. The equation $y^2=4ax$ of a parabola is called the **first standard form,** and $x^2=4ay$ the **second standard form.**]

3. Trace the curve $x^2=-4ay$.

[It is a parabola whose axis runs in the −ve direction of the y-axis.

Note. Downward parabola. Since the parabola $x^2=-4ay$ lies wholly *downwards* of the x-axis, it is called the **downward parabola.**



$x^2=-4ay$

Cor. To find the focus and the directrix of the parabola $x^2=-4ay$. The focus lies on the axis, and AS$=a$.

∴ S is **(0, −a)**.

Again, the directrix ZM is ⊥ to the axis, AZ$=a$ on the opposite side of A as S, ∴ the equation of the directrix is **y=a.**

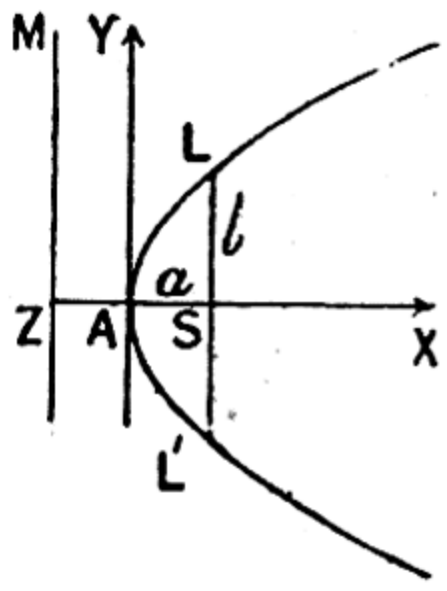
Note. Important. *To write down the standard results, the student should fix in his memory the positions (Figs.) of the parabolas with respect to the co-ordinate axes.*]

[Aid to memory.

$y^2=4ax$ *(right-handed parabola)*	$x^2=4ay$ *(upward parabola)*
$y^2=-4ax$ *(left-handed parabola)*	$x^2=-4ay$ *(downward parabola).*]

97. Latus rectum. Def. *The* **latus rectum** *of a parabola (or conic) is the chord through the focus perpendicular to the axis.*

98. Length of the latus rectum. *To find the length of the latus rectum of the parabola* $y^2=4ax$.



The equation of the parabola is $y^2=4ax$.
Let LSL′ be the latus rectum, and SL$=l$.
Then the pt. L(a, l) lies on the parabola
$\therefore\ l^2=4a.a=4a^2,\ \therefore\ l=2a.$
$\therefore$ LSL′$=2l=2(2a)=$**4a**.

[**N. B.** *The latus rectum of each of the parabolas* $y^2=4ax,\ y^2=-4ax\ ;\ x^2=4ay,\ x^2=-4ay$ *is* **4a**.

The result may be proved in every case by drawing the Fig. and proceeding as in Art. 98.]

Cor. 1. *The equation of the latus rectum is* **x=a** [‖ to y-axis at a distance a (Art. 16)].

Cor. 2. *The co-ordinates of L (the positive end of the latus rectum) are* **(a, 2a)**, *and those of L′ (the negative end) are* **(a, −2a)**.

EXAMPLES

1. In each of the parabolas $(i)\ y^2=8x\ ;\ (ii)\ 3y^2=8x\ ;\ (iii)\ y^2=-4x$, **find** (a) **the length of the latus rectum,** (b) **the co-ordinates of the focus,** (c) **the equation of the directrix.** [*C. U.*]

2. The parabola $y^2=4px$ goes through the point $(3, -2)$. Obtain the length of the latus rectum and the co-ordinates of the focus of the parabola. [*C. U. 1934*]

3. In each of the parabolas $(i)\ 3x^2=4y\ ;\ (ii)\ x^2=-12y$, find (a) the length of the latus rectum, (b) the co-ordinates of the focus, (c) the equation of the directrix.

4. Find the equation of the latus rectum of the parabola $y^2=x$ and the co-ordinates of (i) its positive end, (ii) its negative end.

SECTION II

TANGENTS AND NORMALS

99. Equation of the tangent. To find the equation of the tangent at the point (x_1, y_1) **of the parabola** $y^2=4ax$.*

[**Method of chord.**] The equation of the parabola is $y^2=4ax$.
Let (x_2, y_2) be any other pt. on the parabola.
(i) The equation of the line passing thro′ (x_1, y_1), (x_2, y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}\ (x-x_1)\ \ldots(1)$$

(ii) [To find the value of $\dfrac{y_2-y_1}{x_2-x_1}$.]

$\because\ (x_1, y_1), (x_2, y_2)$ lie on the parabola,
$\therefore\quad y_1^2=4ax_1\ \ldots(2)$
$\qquad\quad y_2^2=4ax_2\ \ldots(3)$
Subtracting (2) from (3), $y_2^2-y_1^2=4a(x_2-x_1)$.

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*[*P(P). U. 1956S*]

Factorising, $(y_2-y_1)(y_2+y_1)=4a(x_2-x_1)$

$\therefore \quad \dfrac{y_2-y_1}{x_2-x_1}=\dfrac{4a}{y_2+y_1}.$

(*iii*) Substituting this value of $\dfrac{y_2-y_1}{x_2-x_1}$ in (1), *the equation of the chord thro'* (x_1, y_1), (x_2, y_2) is $y-y_1=\dfrac{4a}{y_2+y_1}(x-x_1)$ ...(4)

(*iv*) Putting $x_2=x_1$, $y_2=y_1$, in (4), the equation of the tangent at (x_1, y_1) is $y-y_1=\dfrac{4a}{2y_1}(x-x_1)$ [Cancel 2]

or $yy_1-y_1^2=2ax-2ax_1$

or $yy_1=2ax+y_1^2-2ax_1$ [But $y_1^2=4ax_1$, from (2)]

$=2ax+4ax_1-2ax_1$

$=2ax+2ax_1=2a(x+x_1)$

or $\mathbf{yy_1=2a(x+x_1)}$, which is the required equation.

[**N.B.** *The equation of the tangent at the point* (x_1, y_1) *of the parabola* $y^2=4ax$ *is* $\mathbf{yy_1=2a(x+x_1)}$.]

[**Rule to find the equation of the tangent at the point** (x_1, y_1) **of a parabola (equation in the standard form) :** *See Rule of Art. 63.*]

[**Rule to write down the equation of the tangent at the point** (x_1, y_1) **of a parabola (equation in the standard form) :**

See Rule of Art. 64.]

Cor. The tangent at the vertex of a parabola is perpendicular to the axis.

Proof. The equation of the tangent at (0, 0) is

$y(0)=2a(x+0)$ [$yy_1=2a(x+x_1)$]

or $x=0$, which is the y-axis, $\perp$ to the x-axis, the axis of the parabola.

Example. Find the equation of the tangent at the point (3, −6) of the parabola $y^2=12x$.

The equation of the parabola is $y^2=12x$.

$\therefore$ the equation of the tangent at (3, −6) is [Art. 64, Rule*]

[Here $x_1=3$, $y_1=-6$]

$y(-6)=12.\ \frac{1}{2}(x+3)$

or $-y=x+3$ or $x+y+3=0$.

[**Check.** The equation of the tangent is satisfied by the co-ordinates of the given point, thus $3+(-6)+3=0$ or $0=0$.]

EXAMPLES

1. Write down the equation of the tangent at the point (1, −3) of the parabola $y^2=9x$.

2. Show that the ordinate of the point of intersection of two tangents to a parabola is the arithmetic mean between the ordinates of the points of contact. [*P (P). U. 1956S*]

[**Sol.** Let the equation of the parabola be $y^2=4ax$. Then the equations of the tangents at (x_1, y_1), (x_2, y_2) are

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***Explanation.** In the equation of the parabola $y^2=12x$, change y^2 to yy_1, *i.e.*, $y(-6)$, x to $\frac{1}{2}(x+x_1)$, *i.e.*, $\frac{1}{2}(x+3)$.

$$yy_1=2a(x+x_1) \quad ...(1)$$
$$yy_2=2a(x+x_2) \quad ...(2)$$

To find the ordinate of the pt. of intersection, *i.e.*, y, subtracting (2) from (1),

$$y(y_1-y_2)=2a(x_1-x_2) \qquad \left[\text{But } y_1^2=4ax_1, \quad \therefore\ x_1=\frac{y_1^2}{4a}\right.$$

$$\left.y_2^2=4ax_2, \quad \therefore\ x_2=\frac{y_2^2}{4a}\right]$$

$$=2a\left(\frac{y_1^2}{4a}-\frac{y_2^2}{4a}\right)=\frac{y_1^2-y_2^2}{2}=\frac{(y_1+y_2)(y_1-y_2)}{2}$$

[Cancel y_1-y_2]

$$\therefore \qquad y=\frac{y_1+y_2}{2}. \quad]$$

100. Equation of the normal. To find the equation of the normal at the point (x_1, y_1) of the parabola $y^2=4ax$.*

The equation of the parabola is $y^2=4ax$.

(*i*) The equation of the tangent at (x_1, y_1) is $yy_1=2a(x+x_1)$. [Art. 99]

$\therefore$ the slope of the tangent $=\frac{2a}{y_1}$.

(*ii*) $\therefore$ the slope of the normal $=-\frac{y_1}{2a}$. [−ve reciprocal]

(*iii*) $\therefore$ the equation of the normal at (x_1, y_1) is

$$y-y_1=-\frac{y_1}{2a}(x-x_1). \qquad [\ y-y_1=m(x-x_1)\ ]$$

Example. *Find the equation of the normal at the point (1, −3) of the parabola $y^2=9x$.*

Method I. The equation of the parabola is $y^2=9x$.

(*i*) The equation of the tangent at $(1, -3)$ is

$$y(-3)=9.\tfrac{1}{2}(x+1) \qquad [\text{ Cancel 3 }]$$

or $$3x+2y+3=0.$$

$\therefore$ the slope of the tangent $=-\frac{3}{2}$.

(*ii*) $\therefore$ the slope of the normal $=\frac{2}{3}$. [−ve reciprocal]

(*iii*) $\therefore$ the equation of the normal at $(1, -3)$ is

$$y-(-3)=\tfrac{2}{3}(x-1) \qquad [\ y-y_1=m(x-x_1)\ ]$$

or $$3y+9=2x-2 \text{ or } 2x-3y-11=0.$$

Method II. **(Without finding the equation of the tangent.)** The equation of the parabola is $y^2=9x$. [Compare with $y^2=4ax$]

Here $4a=9$, $\therefore\ a=\frac{9}{4}$,

$\therefore$ the equation of the normal at $(1, -3)$ is [Here $x_1=1$, $y_1=-3$]

$$y+3=-\frac{-3}{2\left(\frac{9}{4}\right)}(x-1) \qquad \left[y-y_1=-\frac{y_1}{2a}(x-x_1)\ \right]$$

$$=\tfrac{2}{3}(x-1)$$

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*[*P. U. 1950*]

or $3y+9=2x-2$ or $2x-3y-11=0$.

EXAMPLES

1. Find

(*i*) the equation of the tangent at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ of the parabola $y^2=4ax$;

(*ii*) the equation of the normal to the parabola $y^2=4ax$ at the point $(am^2, -2am)$, what is m ? [*C. U. 1950*]

2. Find the equations of the tangent and normal to $y^2=6x$ at the point $\left(\frac{8}{3}, 4\right)$. [*P. U.*]

3. Write down the equations to the tangent and normal at the point of the parabola $y^2=8x$ whose ordinate is 4.

4. Find the equations of the tangents and the normals to the parabola $y^2=4ax$ at the ends of its latus rectum. [*A. U. 1942*]

5. Write down the equations to the tangents and normals to the parabola $y^2=12x$ at the ends of the latus rectum.

6. Show that the straight line $y=x-3a$ is a normal to the parabola $y^2=4ax$ at the point $(a, -2a)$. [*P. U.*]

101. Equation of the normal in terms of its slope. To find the equation of the normal to the parabola $y^2=4ax$ in the form

$$y=mx-2am-am^3*.$$

The equation of the parabola is $y^2=4ax$.

The equation of the normal at (x_1, y_1) is

$$y-y_1=-\frac{y_1}{2a}(x-x_1) \quad \ldots(1)$$ [Art. 100]

Put $-\frac{y_1}{2a}=m$†, $\therefore\ y_1=-2am$.

[To find the value of x_1.]

$\because$ (x_1, y_1) lies on the parabola

$\therefore\quad y_1^2=4ax_1$

or $(-2am)^2=4ax_1$ or $4a^2m^2=4ax_1$, $\therefore\ x_1=am^2$.

Substituting these values of x_1, y_1 in (1),

$$y+2am=-\frac{-2am}{2a}(x-am^2)=mx-am^3,$$

or $y=mx-2am-am^3$, which is the required equation.

Cor. 1. To find the foot of the normal.* The foot of the normal is (x_1, y_1), *i.e.*, $(am^2, -2am)$.

[N.B. *If m is the slope of a normal to the parabola $y^2=4ax$, the equation of the normal is* $y=mx-2am-am^3$, *and the foot of the normal is* $(am^2, -2am)$.**]**

Cor. 2. Any normal in the m-form. *The equation of any normal*

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*[*P(P). U. 1956*]

†**Why this step.** $\because$ the slope of the normal in the required form ($y=mx-2am-am^3$) is m, $\therefore$ we put the slope $-\frac{y_1}{2a}=m$.

to the parabola $y^2=4ax$, in the m-form, is

$$\mathbf{y=mx-2am-am^3.}$$

Example. Find the equation of the normal to the parabola $y^2=8x$ which makes an angle of 45° with the axis ; find also the foot of the normal. [*C. U.*]

(*i*) The equation of the parabola is $y^2=8x$. [Compare with $y^2=4ax$]

Here $4a=8$, $\therefore\ a=2$, $m=\tan 45°=1$,

$\therefore$ the equation of the normal is $y=1.x-2.2.1-2.1^3$ [$y=mx-2am-am^3$]

$$=x-6$$

or $$x-y-6=0.$$

(*ii*) The foot of the normal is $(2.1^2, -2.2.1)$ [$am^2, -2am$]

or $(2, -4)$.

EXAMPLE

(*a*) **Find the equation of the normal to the parabola $y^2=4ax$ in terms of its slope.** [*B. U. 1938*]

(*b*) (*i*) *Find the equation of the normal to the parabola $y^2=4ax$ which makes an angle of 60° with the axis.*

(*ii*) **Find the point of the parabola $y^2=4ax$ at which the normal is inclined at 30° to the axis.** [*C. U.*]

****102. Note on a cubic equation. Def.** An equation which contains the *cube* of the unknown quantity, but no higher power, is called **a cubic equation** or, more shortly, a **cubic.**

Thus the equation $ax^3+bx^2+cx+d=0$ is a **cubic** in x.

It has three roots x_1, x_2, x_3 (say). We shall remember without proof

that **sum of the roots,** *i.e.*, $\mathbf{x_1+x_2+x_3}\left(=-\frac{\mathbf{b}}{\mathbf{a}}\right)=-\frac{\textbf{coeff. of } \mathbf{x^2}}{\textbf{coeff. of } \mathbf{x^3}}$,

sum of the products of the roots taken two at a time,

i.e., $$x_1x_2+x_2x_3+x_3x_1=\frac{c}{a}=\frac{\text{coeff. of } x}{\text{coeff. of } x^3},$$

product of the roots, *i.e.*, $\mathbf{x_1x_2x_3}\left(=-\frac{\mathbf{d}}{\mathbf{a}}\right)=-\frac{\textbf{constant term}}{\textbf{coeff. of } \mathbf{x^3}}$.

103. Three normals from a point to a parabola. To show that from any point three normals can be drawn to a parabola.*

Let the equation of the parabola be $y^2=4ax$, and let the pt. be (x_1, y_1).

The equation of *any* normal to the parabola, in the *m*-form, is

$$y=mx-2am-am^3.$$ [Art. 101]

If it passes thro' (x_1, y_1),

then $$y_1=mx_1-2am-am^3$$

or $$am^3+m(2a-x_1)+y_1=0,$$

which is a cubic in m giving three values of m.

$\therefore$ from (x_1, y_1) three normals can be drawn to the parabola.

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*[*P. U. 1956*]

EXAMPLES

1. If three normals are drawn to a parabola from a given point, prove that

(*i*) the algebraic sum of their slopes is zero. [*P. U.*]

(*ii*) the algebraic sum of the ordinates of their feet is zero. [*C. U. 1954*]

[**Sol.** Let the equation of the parabola be $y^2=4ax$, and let the pt. be (x_1, y_1).

The equation of *any* normal to the parabola, in the m-form, is $y=mx-2am-am^3$. If it passes thro' (x_1, y_1), then $y_1=mx_1-2am-am^3$ or $am^3+m(2a-x_1)+y_1=0$, which is a cubic in m.

Let m_1, m_2, m_3 be the roots. Then

(*i*) The sum of the slopes

$$=m_1+m_2+m_3=-\frac{\text{coeff. of } m^2}{\text{coeff. of } m^3} \quad \text{[Sum of the roots (Art. 102)]}$$

$$=-\frac{0}{a}=0.$$

(*ii*) The sum of the ordinates of the feet

$=(-2am_1)+(-2am_2)+(-2am_3)$

$=-2a(m_1+m_2+m_3)=-2a(0)$ [As in part (*i*)]

$=0.$]

2. Find the locus of the point of intersection of two normals to a parabola which are at right angles to one another. [*P(P). U. 1950*]

[**Sol.** Let the equation of the parabola be $y^2=4ax$.

The equation of *any* normal to the parabola, in the m-form, is $y=mx-2am-am^3$. If it passes thro' (x_1, y_1), then $y_1=mx_1-2am-am^3$ or $am^3+m(2a-x_1)+y_1=0$ ...(1), which is a cubic in m.

Let m_1, m_2, m_3 be the roots.

If two normals are $\perp$, $m_1m_2=-1$ ...(2)

But from (1), $m_1m_2m_3=-\dfrac{\text{constant term}}{\text{coeff. of } m^3}$

[Product of the roots (Art. 102)]

$$=-\frac{y_1}{a}$$

$\therefore$ from (2), $(-1)m_3=-\dfrac{y_1}{a}$, $\therefore\ m_3=\dfrac{y_1}{a}$.

But m_3 is a root of (1),

$\therefore$ substituting this value of $m\left(=\dfrac{y_1}{a}\right)$ in (1),

$a\dfrac{y_1^3}{a^3}+\dfrac{y_1}{a}(2a-x_1)+y_1=0$ or, cancelling y_1, $\dfrac{y_1^2}{a^2}+\dfrac{(2a-x_1)}{a}+1=0$,

or $y_1^2+a(2a-x_1)+a^2=0$ or $y_1^2+3a^2-ax_1=0$. $\therefore$ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y)], $y^2+3a^2-ax=0$ or $y^2=a(x-3a)$.]

3. Find the locus of intersection of normals to a parabola inclined at complementary angles to the axis. [*P. U.*]

SECTION III

A PARABOLA AND A LINE

104. Points of intersection. To find the points of intersection of the line $y=mx+c$ with the parabola $y^2=4ax$.

The equation of the line is $y=mx+c$ ...(1)

,, ,, parabola is $y^2=4ax$ ...(2)

(*i*) Substituting the value of y from (1) in (2),

$$(mx+c)^2=4ax$$

or $$m^2x^2+2mcx+c^2=4ax$$

or $$m^2x^2+2x(mc-2a)+c^2=0,$$

which is a quadratic in x giving two values of x.

(*ii*) Substituting these values of x one by one in (1), we get the corresponding values of y.

These corresponding values of x and y are the co-ordinates of the required pts. of intersection.

[Rule to find the points of intersection of a line and a parabola : *See Rule of Art. 69.*]

EXAMPLES

1. Find the points where the line $y=3x-a$ meets the parabola $y^2=4ax$.

2. Find the co-ordinates of the points where the line $x-5y+6=0$ meets the parabola $y^2=x$. [*C. U. 1944*]

3. Show that the normal to the parabola $y^2=8x$ at (2, 4) meets the parabola again at (18, −12). [*P(P). U. 1951*]

105. Length of the intercept. To find the length of the intercept made by the parabola $y^2=4ax$ on the line $y=mx+c$.*

The equation of the line is $y=mx+c$ ...(1)

,, ,, parabola is $y^2=4ax$ ...(2)

(*i*) Substituting the value of y from (1) in (2),

$$(mx+c)^2=4ax$$

or $$m^2x^2+2mcx+c^2=4ax$$

or $$m^2x^2+2x(mc-2a)+c^2=0 \quad ...(3)$$

which is a quadratic in x. Let x_1, x_2 be the roots.

[To find the value of x_2-x_1.]

From (3), $x_1+x_2=-\dfrac{2(mc-2a)}{m^2}$, [Sum of the roots]

and $$x_1x_2=\frac{c^2}{m^2}.$$ [Product of the roots]

$$\therefore \quad x_2-x_1=\sqrt{(x_2+x_1)^2-4x_2x_1}=\sqrt{\frac{4(mc-2a)^2}{m^4}-\frac{4c^2}{m^2}}$$

$$\therefore\ x_2-x_1=\sqrt{\frac{4(mc-2a)^2-4m^2c^2}{m^4}}=\frac{2}{m^2}\sqrt{(mc-2a)^2-m^2c^2}$$

$$=\frac{2}{m^2}\sqrt{m^2c^2-4mca+4a^2-m^2c^2}$$

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*[*N. U. 1953*]

$$= \frac{4}{m^2}\sqrt{a(a-mc)} \quad ...(4)$$

(*ii*) [To find the value of y_2-y_1.]

$\because$ (x_1, y_1), (x_2, y_2) lie on the *line*, [*Note this step*]

$\therefore$ $y_1=mx_1+c$ ...(5)

$y_2=mx_2+c$ ...(6)

Subtracting (5) from (6), $y_2-y_1=m(x_2-x_1)$ ... (7)

(*iii*) Now length of the intercept $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2}$ [From (7)]

$=(x_2-x_1)\sqrt{1+m^2}$

$=\frac{4}{m^2}\sqrt{a(a-mc)}\sqrt{1+m^2}$. [From (4)]

[Rule to find the length of the intercept made by a parabola on the line y=mx+c : *See Rule of Art. 70.*]

Cor. Condition of tangency. To deduce the condition that the line may touch the parabola.*

If the line touches the parabola, the length of the intercept $=0$

i.e., $\frac{4}{m^2}\sqrt{a(a-mc)}\sqrt{1+m^2}=0$

$\therefore$ $a-mc$† $=0$, $\therefore$ $c=\frac{a}{m}$, which is the required condition.

EXAMPLES

1. Find the length of the chord which is normal to the curve $y^2=12x$ at the point (3, 6). [*P. U.*]

[Short-cut (for the length of the intercept) for numerical examples. See Note given in Art. 70.]

**2. Show that the length of the line $2x-2y+1=0$ intercepted by the parabola $y^2=4x$ is 4. [*P (P). U. 1956S*]

[**Hint.** The equation of the line is $y=x+\frac{1}{2}$. [Form $y=mx+c$]

Proceed as in Art. 105.]

106. Condition of tangency. To find the condition that the line y=mx+c may touch the parabola $y^2=4ax$. ‡

[Method of intersections.]

The equation of the line is $y=mx+c$ ... (1)

,, ,, parabola is $y^2=4ax$ ... (2)

(*i*) Substituting the value of y from (1) in (2),

$$(mx+c)^2=4ax$$

or $$m^2x^2+2mcx+c^2=4ax$$

or $$m^2x^2+2x(mc-2a)+c^2=0 \quad ...(3)$$

which is a quadratic in x.

(*ii*) If the line *touches* the parabola, the quadratic (3) has equal roots.

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* [*J. & K. U. 1952*]

‡ [*C. U. 1955*]

† *i.e.*, the factor involving *c*.

$\therefore \quad 4(mc-2a)^2-4m^2c^2=0$ [$b^2-4ac=0$]

[Cancel 4]

or $m^2c^2-4mca+4a^2-m^2c^2=0$ [Cancel $4a$]

or $-mc+a=0$

$\therefore \quad \mathbf{c=\frac{a}{m}}$,

which is the required condition.

[Rule to find the condition that the line $y=mx+c$ may touch a parabola. *See Rule of Art. 71.*]

Cor. 1. To find the equation of a tangent in terms of its slope.*

Substituting the value of $c\left(=\frac{a}{m}\right)$ in (1),

the equation of the tangent is $\mathbf{y=mx+\frac{a}{m}}$ ...(4)

Cor. 2. To find the point of contact.* Substituting the value of $c\left(=\frac{a}{m}\right)$ in (3), $m^2x^2+2x\left(m.\frac{a}{m}-2a\right)+\frac{a^2}{m^2}=0$,

or $m^2x^2-2ax+\frac{a^2}{m^2}=0$ or $\left(mx-\frac{a}{m}\right)^2=0$

or $mx=\frac{a}{m}$, $\therefore \quad x=\frac{a}{m^2}$.

Substituting this value of x in (4), $y=m.\frac{a}{m^2}+\frac{a}{m}=\frac{2a}{m}$.

$\therefore$ the pt. of contact is $\left(\mathbf{\frac{a}{m^2}}, \mathbf{\frac{2a}{m}}\right)$.

[**N.B.** *If m is the slope of a tangent to the parabola $y^2=4ax$, the equation of the tangent is* $\mathbf{y=mx+\frac{a}{m}}$,

and the point of contact is $\left(\mathbf{\frac{a}{m^2}}, \mathbf{\frac{2a}{m}}\right)$.]

[**Aid to memory.** If m is the slope of the tangent,

the point of contact is $\left(\mathbf{\frac{a}{m^2}}, \mathbf{-\frac{2a}{m}}\right)$ ...(1)

If m is the slope of the normal (*i.e.*, negative reciprocal of the slope of the tangent), then changing m to $-\frac{1}{m}$ in (1),

the foot of the normal is $\left\{\frac{a}{\left(-\frac{1}{m}\right)^2}, \frac{2a}{-\frac{1}{m}}\right\}$, *i.e.*, $\mathbf{(am^2, -2am)}$.]

Cor. 3. Any tangent in the m-form. *The equation of any tangent to the parabola $y^2=4ax$, in the m-form, is* $\mathbf{y=mx+\frac{a}{m}}$.

****Note. Another method to find the equation of the tangent to the**

*[*Bar. U. 1953*]

parabola $y^2=4ax$ in the form $y=mx+\frac{a}{m}$ and to find the co-ordinates of the point of contact.

The equation of the parabola is $y^2=4ax$.

The equation of the tangent at (x_1, y_1) is

$$yy_1=2a(x+x_1)$$

or
$$y=\frac{2a}{y_1}x+2a\frac{x_1}{y_1} \quad \ldots (1)$$

Put $\frac{2a}{y_1}=m^*$, $\therefore\ y_1=\frac{2a}{m}$.

[To find the value of x_1.]

$\because$ (x_1, y_1) lies on the parabola

$\therefore\ y_1^2=4ax_1$ or $\frac{4a^2}{m^2}=4ax_1$, $\therefore\ x_1=\frac{a}{m^2}$.

Substituting these values of x_1, y_1 in (1),

$$y=\frac{2a}{\frac{2a}{m}}x+2a\frac{\frac{a}{m^2}}{\frac{2a}{m}}=2a.\frac{m}{2a}x+2a.\frac{a}{m^2}\cdot\frac{m}{2a}$$

or $y=mx+\frac{a}{m}$, which is the required equation.

The pt. of contact is (x_1, y_1), *i.e.*, $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

EXAMPLES

1. Show that the line $2y=4x+a$ touches the parabola $y^2=4ax$. Find the co-ordinates of the point of contact. [*D. U. 1936*]

2. Show that $y=mx+\frac{a}{m}$ touches the parabola $y^2=4ax$, and find the co-ordinates of its point of contact. [*Pesh. U. 1955*]

3. (*i*) **Show that $y=mx+c$ touches the parabola $y^2=4a(x+a)$ if**

$$c=am+\frac{a}{m}.$$

(*ii*) Find the value of a for which the line $x+y+2=0$ touches the parabola $y^2=ax$. Find the co-ordinates of the point of contact. [*P(P). U. 1952*]

4. Condition of tangency. Find the condition that the straight line $lx+my+n=0$, should touch the parabola $y^2=4ax$; and find the co-ordinates of the point of contact. [*J. & K. U. 1951*]

5. Write down the equation of the tangent to the parabola $y^2=x$ which is inclined to the axis of x at an angle of 45°. Find also the co-ordinates of the point of contact. [*P. U. 1949*]

***Why this step.** $\because$ the slope of the tangent in the required form $\left(y=mx+\frac{a}{m}\right)$ is m, $\therefore$ we put the slope $\frac{2a}{y_1}=m$.

[**Sol.** The equation of the parabola is $y^2=x$.
(Compare with $y^2=4ax$)

(*i*) Here $4a=1$, $\therefore$ $a=\frac{1}{4}$; $m=\tan 45°=1$,

$\therefore$ the equation of the tangent is $y=1.x+\frac{\frac{1}{4}}{1}$ $\left[\ y=mx+\frac{a}{m}\ \right]$

or $4x-4y+1=0$.

(*ii*) The pt. of contact is $\left(\frac{\frac{1}{4}}{1^2},\ \frac{2.\frac{1}{4}}{1}\right)$ $\left[\frac{a}{m^2},\ \frac{2a}{m}\right]$

or $(\frac{1}{4}, \frac{1}{2})$.

Or thus: Use the method given in Note at the end of Art. 106. Here $m=\tan 45°=1$.]

6. (*a*) Find the equations of the two tangents to the parabola $y^2=4ax$ which are equally inclined to the co-ordinate axes. [*C. U.*]

(*b*) **A tangent to the parabola $y^2=16x$ makes an angle of 60° with the x-axis. Find its point of contact.** [*P.U. 1931S*]

7. Find the points of the parabola $y^2=4ax$ at which (*i*) the tangent, and (*ii*) the normal is inclined at 30° to the axis.

8. A tangent to the parabola $y^2=4ax$ cuts off equal intercepts on the axes. Find its equation.

9. Write down the equations of the tangents to $y^2=16x$ parallel and perpendicular to $3x-4y+5=0$. Find also their points of contact. [*P. U.*]

SECTION IV

A PARABOLA AND A POINT

107. Test for a point to lie outside, on, or inside a parabola. *To show that the quantity $y_1^2-4ax_1$ is positive, zero, or negative, according as the point (x_1, y_1) lies outside, on, or inside the parabola $y^2=4ax$.*

The equation of the parabola is $y^2=4ax$.

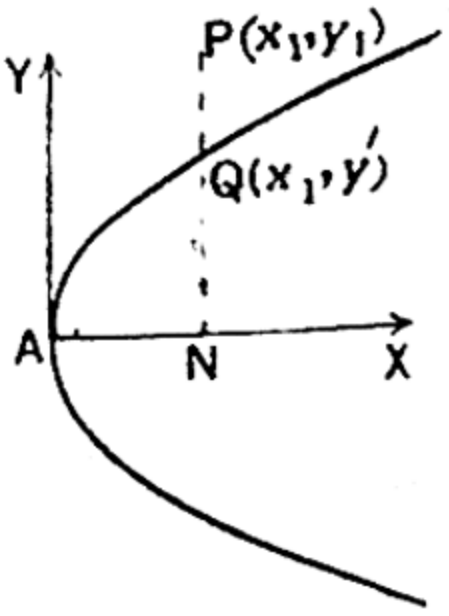
Let P be the pt. (x_1, y_1).

From P draw PN $\perp$ on the x-axis to meet the parabola in Q (x_1, y').

$\because$ Q lies on the parabola

$\therefore$ $y'^2=4ax_1$ or $y'^2-4ax_1=0$ ... (1)

[R. H. S. zero]



Now $y_1^2-4ax_1=y_1^2-4ax_1-0$

[Substitute for 0 from (1)]

$=y_1^2-4ax_1-(y'^2-4ax_1)$

$=y_1^2-y'^2=NP^2-NQ^2$ ... (2)

(*i*) If P lies outside the parabola (as in the Fig.), **NP > NQ,**

$\therefore$ from (2),* $y_1^2-4ax_1$ is +ve.

(*ii*) If P lies on the parabola, NP=NQ,

$\therefore$ from (2), $y_1^2-4ax_1$ is zero.

(*iii*) If P lies inside the parabola, NP < NQ,

$\therefore$ from (2), $y_1^2-4ax_1$ is −ve.

---

* $\because$ R. H. S. is +ve, $\therefore$ L. H. S. is also +ve.

[Rule to find whether a point lies outside, on, or inside a parabola (equation in the standard form) : *See Rule of Art. 72*]

EXAMPLE

Does the point $(2, -3)$ lie outside, on, or inside the parabola (*i*) $y^2=5x$; (*ii*) $y^2-4x=0$; (*iii*) $2y^2=9x$?

108. Two tangents from a point. To show that from any point two tangents can be drawn to a parabola, and they are real, coincident, or imaginary according as the point is outside, on, or inside the parabola.*

Let the equation of the parabola be $y^2=4ax$, and let the pt. be (x_1, y_1).

(*i*) The equation of *any* tangent to the parabola, in the *m*-form, is $y=mx+\frac{a}{m}$. [Art. 106, Cor. 3]

(*ii*) If it passes thro' (x_1, y_1), then

$$y_1=mx_1+\frac{a}{m} \quad \text{or } m^2x_1-my_1+a=0 \quad \ldots(1)$$

which is a quadratic in m giving two values of m.

$\therefore$ from (x_1, y_1) two tangents can be drawn to the parabola.

(*iii*) The tangents are real, coincident, or imaginary according as the roots of the quadratic (1) are real, equal, or imaginary, *i.e.*, according as $y_1^2-4ax_1$ is +ve, 0, or −ve [b^2-4ac is +ve, 0, or −ve] *i.e.*, according as the pt. (x_1, y_1) lies outside, on, or inside the parabola. [Art. 107]

Example. Find the equations of the two tangents to the parabola $y^2=6x$ which pass through the point $\left(\frac{3}{2}, 5\right)$. [*P.U. 1936S*]

The equation of the parabola is $y^2=6x$. [Compare with $y^2=4ax$]

Here $4a=6$, $\therefore\ a=\frac{3}{2}$.

(*i*) The equation of *any* tangent to the parabola, in the *m*-form, is

$$y=mx+\frac{\frac{3}{2}}{m} \quad \ldots(1) \qquad \left[\ y=mx+\frac{a}{m} \text{ (Art. 106, Cor. 3)}\ \right]$$

(*ii*) If it passes thro' $\left(\frac{3}{2}, 5\right)$, then $5=m\left(\frac{3}{2}\right)+\frac{3}{2m}$

or $10m=3m^2+3$ or $3m^2-10m+3=0$

$$\therefore\ m=\frac{10\pm\sqrt{100-36}}{6}=\frac{10\pm8}{6}=\frac{18}{6}, \frac{2}{6}$$

$$=3, \tfrac{1}{3}.$$

---

* [*C. U. 1951*]

(*iii*)* ∴ the equations of the tangents from $\left(\frac{3}{2}, 5\right)$ are

(I) $y-5=3\left(x-\frac{3}{2}\right)$ [$y-y_1=m(x-x_1)$]

$=3x-\frac{9}{2}$

or $2y-10=6x-9$ or $6x-2y+1=0$.

(II) $y-5=\frac{1}{3}\left(x-\frac{3}{2}\right)$

$=\frac{x}{3}-\frac{1}{2}$

or $6y-30=2x-3$

or $2x-6y+27=0$.

[**Rule to find the equations of the tangents drawn from a given point to a parabola (equation in the standard form) :** *See Rule of solved Ex., Art. 73.*]

EXAMPLES

1. Find the equations of the tangents to the parabola $y^2=9x$ which go through the point (4, 10). [*M.B.*]

2. Find the angle between two tangents drawn from the point (1, 4) to the parabola $y^2=12x$. [*P(P). U. 1955*]

109. To find the locus of the point of intersection of two perpendicular tangents to a parabola.†

Let the equation of the parabola be $y^2=4ax$.

(*i*) The equation of *any* tangent to the parabola, in the *m*-form, is $y=mx+\frac{a}{m}$. [Art. 106, Cor. 3]

(*ii*) If it passes thro' (x_1, y_1), then

$y_1=mx_1+\frac{a}{m}$ or $m^2x_1-my_1+a=0$ ...(1)

which is a quadratic in m. Let m_1, m_2 be the roots.

(*iii*) If the tangents are $\perp$, $m_1m_2=-1$

∴ from (1), $\frac{a}{x_1}=-1$ or $x_1=-a$

∴ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y) (Art. 74)] $x=-a$, which is the directrix.

[**Rule to find the locus of the point of intersection of two perpendicular tangents to a parabola (equation in the standard form) :** *See Rule of Art. 75.*]

---

* **Or thus :** (*iii*) Substitute these values of m in (1), which method does *not* apply in the case of the circle or the ellipse or the hyperbola. (See Caution in solved Ex., Art. 73)

† [*P(P). U. 1954*]

EXAMPLE

From an external point P tangents are drawn to the parabola $y^2=4ax$; find the equation to the locus of P when the tangents make angles θ_1 and θ_2 with the x-axis such that

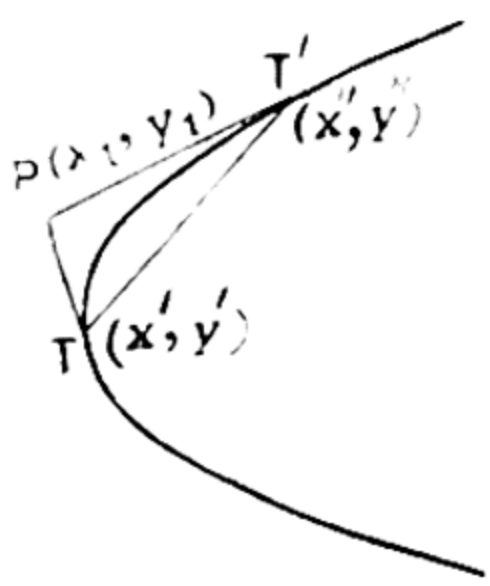
(*i*) $\tan\theta_1+\tan\theta_2=k$; (*ii*) $\tan\theta_1\tan\theta_2=k$;
(*iii*) $\tan^2\theta_1+\tan^2\theta_2=k$; [*M. U.*] (*iv*) $\cot\theta_1+\cot\theta_2=k$.

Chord of contact

110. Equation of the chord of contact. To find the equation of the chord of contact of tangents from (x_1, y_1) to the parabola $y^2=4ax$.*

The equation of the parabola is $y^2=4ax$.

Let P be the pt. (x_1, y_1), and $T(x', y')$, $T'(x'', y'')$, the pts. of contact of the tangents from P to the parabola.



(*i*) Then the equations of the tangents at T, T' are

$$yy' = 2a(x+x') \quad \ldots (1)$$
$$yy'' = 2a(x+x'') \quad \ldots (2)$$

(*ii*) $\because$ both these tangents pass thro' P,

$$\therefore\ y_1y' = 2a(x_1+x') \quad \ldots (3)$$
$$y_1y'' = 2a(x_1+x'') \quad \ldots (4)$$

(*iii*) $\therefore$ from (3) and (4), $T(x', y')$, $T'(x'', y'')$ lie on the line $y_1y=2a(x_1+x)$,† *i.e.*, $\mathbf{yy_1=2a(x+x_1)}$, which is, therefore, the required equation of the chord of contact. [Def. (Art. 76)]

[**N.B.** *The equation of the chord of contact of tangents from* (x_1, y_1) *to the parabola* $y^2=4ax$ *is* $\mathbf{yy_1=2a(x+x_1)}$.]

EXAMPLES

1. If the point (x_1, y_1) lies on the directrix of the parabola, show that its chord of contact passes through the focus. [*P. U 1955*]

[**Hint.** Let the equation of the parabola be $y^2=4ax$. Then the equation of the directrix is $x=-a$. If (x_1, y_1) ...(1) lies on it, then $x_1=-a$, $\therefore$ from (1), the pt. is $(-a, y_1)$. Find the equation of its chord of contact w.r.t. the parabola.]

2. Prove that in a parabola the chord of contact of tangents at right angles passes through the focus. [*P. U. 1937*]

3. In the parabola $y^2=4x$, find the points of contact of tangents from (2, 3). [*M.U.*]

Pole and polar

111. Equation of the polar. To find the equation of the polar of the point (x_1, y_1) with respect to the parabola $y^2=4ax$.‡

The equation of the parabola is $y^2=4ax$.

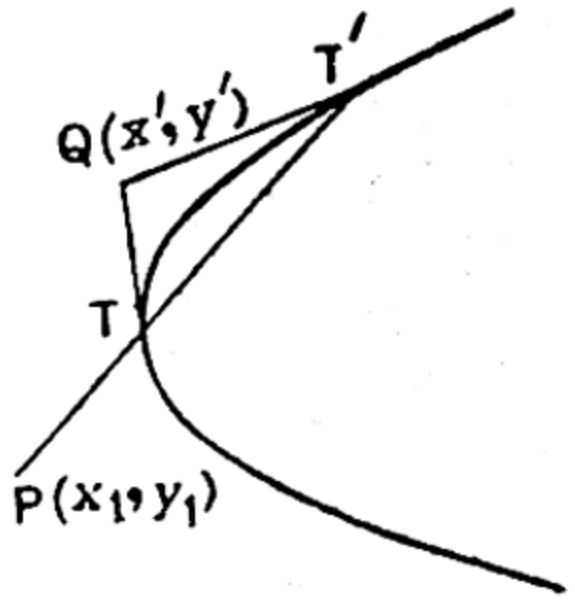
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*[*P. U. 1955*]

†**How to write down this equation.** In the equations (3) and (4) change (x', y') and (x'', y'') to (x, y).

‡[*P. U. 1953*]

Let P be the pt. (x_1, y_1) and PTT′ *any* line thro' P to meet the parabola in T, T′. Let the tangents at T, T′ meet in $Q(x', y')$.



(*i*) The equation of TT′, the chord of contact of tangents from Q, is

$$yy'=2a(x+x'). \qquad \text{[Art. 110]}$$

(*ii*) ∵ it passes thro' P,

$$\therefore\ y_1y'=2a(x_1+x').$$

(*iii*) ∴ the locus of $Q(x', y')$ is

[changing (x', y') to (x, y)]

$$y_1y=2a(x_1+x),$$

i.e., $\mathbf{yy_1=2a(x+x_1)}$, which is the required equation of the polar of P [Def. (Art. 78)], and is clearly a st. line.

[**N.B.** *The equation of the polar of* (x_1, y_1) *w.r.t. the parabola* $y^2=4ax$ *is* $\mathbf{yy_1=2a(x+x_1).}$]

[**Aid to memory.** See Art. 79, **Aid to memory.**]

EXAMPLES

1. (*a*) Define the polar of a point w.r.t. a parabola. [*P(P). U. 1956*]

(*b*) **Show that the polar of the focus with respect to a parabola is the directrix.** [*P(P). U. 1956*]

[Sol. (*b*) Let the equation of the parabola be $y^2=4ax$. Then the equation of the polar of the focus $(a, 0)$ is $y(0)=2a(x+a)$ $[yy_1=2a(x+x_1)]$ or $x+a=0$ or $x=-a$, which is the directrix.]

2. Prove that the polars of all points on the directrix of a parabola pass through the focus. [*P. U. 1953*]

112. Pole of a given line. To find the pole of the line $Ax+By+C=0$ with respect to the parabola $y^2=4ax$.

The equation of the parabola is $y^2=4ax$.

Let (x_1, y_1) be the required pole.

(*i*) The equation of the polar of (x_1, y_1) is

$$yy_1=2a(x+x_1) \text{ or } 2ax-yy_1+2ax_1=0 \quad \ldots(1)$$

(*ii*) ∵ it is the same as the equation of the given line

$$Ax+By+C=0 \quad \ldots(2)$$

(*iii*) ∴ comparing coefficients in (1) and (2), $\dfrac{2a}{A}=\dfrac{-y_1}{B}=\dfrac{2ax_1}{C}$

$$\therefore\ x_1=\frac{C}{A} \text{ (from the first and third members),}$$

$$y_1=-2\frac{B}{A}a \text{ (from the first and second members),}$$

$$\therefore \text{ the required pole is } \left(\frac{C}{A}, -2\frac{B}{A}a\right).$$

[**Rule to find the pole of a given line with respect to a parabola :** *See Rule of Art. 80.*]

EXAMPLES

1. Find the pole of the line $2x-3y=2a$ with respect to the parabola $y^2=4ax$. *[P(P). U.]*

2. Find the pole of the line $y=2x$ with respect to the parabola $y^2=2x$. *[P. U. 1950]*

3. Find the co-ordinates of the point of intersection of tangents drawn to the parabola $y^2=4ax$ at the points where it is cut by the line $Ax+By+C=0$. *[A. U. 1939]*

Properties of pole and polar

113. Reciprocal property. If the polar of a point P with respect to a parabola passes through a point Q, then the polar of Q passes through P.

Let the equation of the parabola be $y^2=4ax$. Let P, Q be the pts. (x_1, y_1), (x_2, y_2). Then the equation of the polar of P is $yy_1=2a(x+x_1)$.

If it passes thro' Q, then $y_2y_1=2a(x_2+x_1)$ ...(1)

Now the equation of the polar of Q is $yy_2=2a(x+x_2)$ ...(2)

Substituting the co-ordinates of P in (2), we get $y_1y_2=2a(x_1+x_2)$, which is true from (1). $\therefore$ the polar of Q passes thro' P.

EXAMPLE

If the polars of two points P and Q with respect to a parabola meet in R, then R is the pole of the line PQ.

Use of parameter

114. To express the co-ordinates of any point on the parabola $y^2=4ax$ in terms of one variable (called the *parameter*).*

The equation of the parabola is $y^2=4ax$.

Substituting the co-ordinates $x=at^2$, $y=2at$ in the equation of the parabola, we get $(2at)^2=4a.at^2$ or $4a^2t^2=4a^2t^2$, which is true.

$\therefore$ **$x=at^2$, $y=2at$** are the required co-ordinates.

Cor. Any point on the parabola. *Any point on the parabola $y^2=4ax$ is* **$(at^2, 2at)$.**

Note. Since the position of the point $P(at^2, 2at)$ is known when the value of t is given, hence the

Def. *The point whose co-ordinates are $(at^2, 2at)$ is, for shortness, called the* **"point t."**

Note. The co-ordinates $x=\frac{a}{m^2}$, $y=\frac{2a}{m}$ [m being the slope of the tangent], and $x=am^2$, $y=-2am$ [m being the slope of the normal], also satisfy the equation of the parabola $y^2=4ax$, and may, therefore, be taken as the co-ordinates of any point on the parabola, in terms of one variable m, but the co-ordinates $x=at^2$, $y=2at$ of Art. 114, avoiding both fractions and negative sign, are preferable.

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*[*P. U. 1956*]

EXAMPLES

1. Show that the point $(at^2, 2at)$ always lies on the parabola $y^2=4ax$. [*Pesh. U. 1954*]

2. Show that the locus of the middle points of chords of the parabola $y^2=4ax$ drawn through the vertex is the parabola $y^2=2ax$. [*R. B. 1950*]

3. PQ is a double ordinate of a parabola. Find the locus of its points of trisection.

4. If t_1, t_2, t_3 are three points on the parabola $y^2=4ax$, show that the area of the triangle formed by them is $a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$.
[*Cf.* Art. 9, Ex. 2, (*b*), (*ii*).]

5. Prove that the point $P\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ lies on the parabola $y^2=4ax$ for all values of m. What is the geometrical significance of m? Find the equation of the tangent at P to the parabola. [*B. U. 1943*]

115. To find the equation of the straight line joining two points, t_1 and t_2, on the parabola $y^2=4ax$.*

The equation of the parabola is $y^2=4ax$.

The co-ordinates of the pts. t_1, t_2

$$\begin{matrix}(1) & (2)\\ (at_1^2, 2at_1), & (at_2^2, 2at_2)\end{matrix}$$ are [Def. (Art. 114)]

$\therefore$ the equation of the line passing thro' them is

$$y-2at_1=\frac{2at_2-2at_1}{at_2^2-at_1^2}(x-at_1^2) \qquad \left[\, y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \,\right]$$

[Cancel a from the num. and denom.]

$$=\frac{2(t_2-t_1)(x-at_1^2)}{(t_2+t_1)(t_2-t_1)}$$

[Cancel (t_2-t_1) from the num. and denom.]

$$=\frac{2}{(t_2+t_1)}(x-at_1^2)$$

or $y(t_2+t_1)-2at_1(t_2+t_1)=2(x-at_1^2)$

or $y(t_2+t_1)-2at_1t_2-2at_1^2=2x-2at_1^2$

or $y(t_1+t_2)=2x+2at_1t_2$ ...(1), which is the required equation.

Cor. *To deduce the equation of the tangent at the point t_1.* *

Putting $t_2=t_1$ in (1), the equation of the tangent at the pt. t_1 is

$$y.2t_1=2x+2at_1^2$$ [Cancel 2]

or $x-t_1y+at_1^2=0$.

EXAMPLES

1. (*i*) **Show that the equation of the tangent to the parabola $y^2=4ax$ at the point $(at^2, 2at)$ is $x-yt+at^2=0$.** [*D. U. 1931*]

(*ii*) **Find the equation of the normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.**

Hence or otherwise show that from any point three normals can be drawn to a parabola. [*P. U. 1956*]

[**Note. Geometrical meaning of t.** From Ex. 1, (*i*),

the slope of the tangent $=\frac{1}{t}$

*[*Pesh. U. 1954*]

∴ the parameter t is the reciprocal of the slope of the tangent at $P(at^2, 2at)$.]

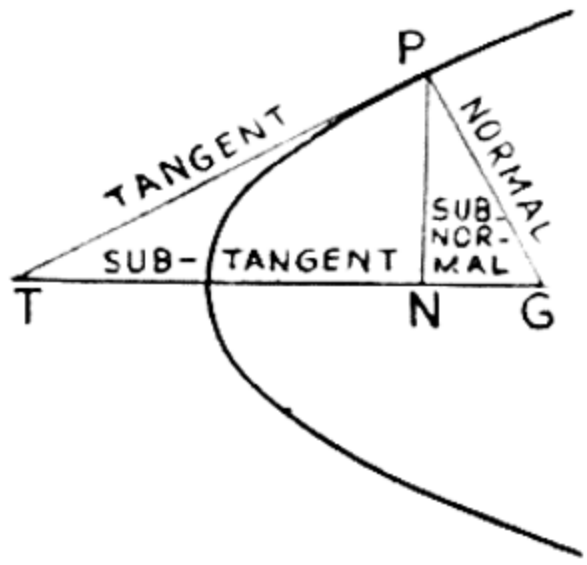
2. Find the point of intersection of the tangents at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ of the parabola $y^2=4ax$. [*P. U. 1944*]

****3. Find the condition that $y=mx+c$ be a normal to the parabola $y^2=4ax$.** [*A. U. 1948*]

SECTION V

GEOMETRICAL PROPERTIES

116. Subtangent and subnormal. Defs. *Let the tangent and normal at any point P of a parabola (or conic) meet the axis in T and G respectively, and let NP be the ordinate of P, then*



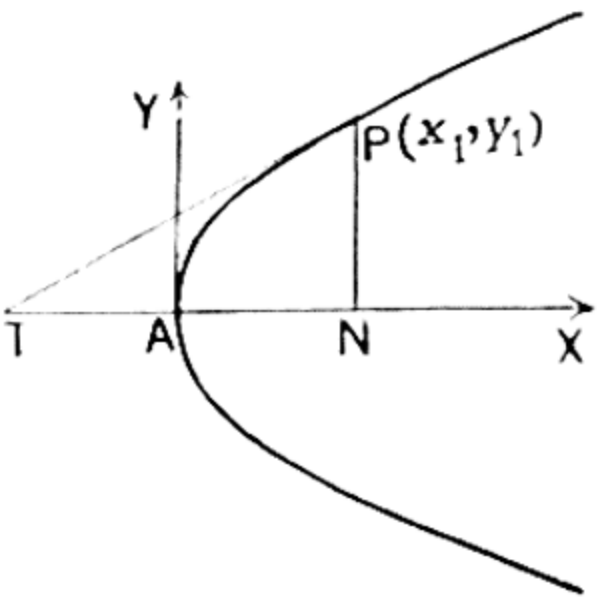
(*i*) *TN is called the* **subtangent** *at P, and*
(*ii*) *NG is called the* **subnormal** *at P.*

[**Aid to memory.** *The* **subtangent** *is the projection of the* **tangent** *on the axis and* **subnormal** *the projection of the* **normal.**]

117. (*a*) Subtangent. The subtangent at any point of a parabola is bisected at the vertex.

(AT=AN)*

Let the equation of the parabola be $y^2=4ax$. Let $P(x_1, y_1)$ be *any* pt. on the parabola and the tangent at P meet the axis in T. From P draw PN ⊥ on the axis.



The equation of the tangent at P is

$$yy_1=2a(x+x_1) \quad ...(1)$$

It meets the x-axis where, putting $y=0$ in (1),

$$0=2a(x+x_1)$$

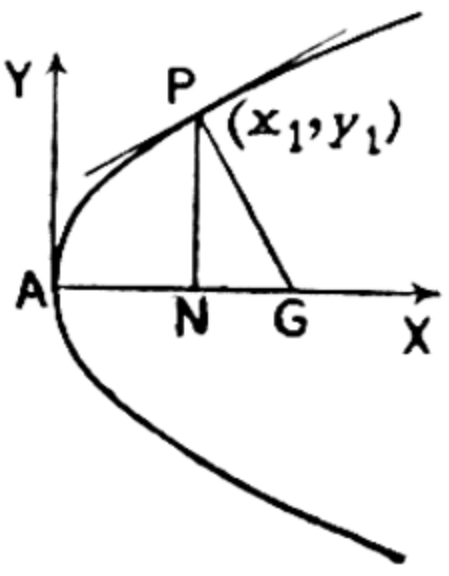
or $x=-x_1$ or AT=−AN†

∴ *changing the sign,* AT *in magnitude*=AN.

117. (*b*) Subnormal. The subnormal at any point of a parabola is constant and equal to half the latus rectum.‡

(NG=2a)

Let the equation of the parabola be $y^2=4ax$. Let $P(x_1, y_1)$ be *any* pt. on the parabola and the normal at P meet the axis in G. From P draw PN ⊥ on the axis.



The equation of the normal at P is

$$y-y_1=-\frac{y_1}{2a}(x-x_1) \quad ...(1) \qquad \text{[Art. 100]}$$

It meets the x-axis where, putting $y=0$ in (1),

$$-y_1=-\frac{y_1}{2a}(x-x_1) \qquad \text{[Cancel } -y_1\text{]}$$

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*[*P. U. 1946*]
†The negative sign shows that AT and AN are drawn in opposite directions.
‡[*P. U. 1947*]

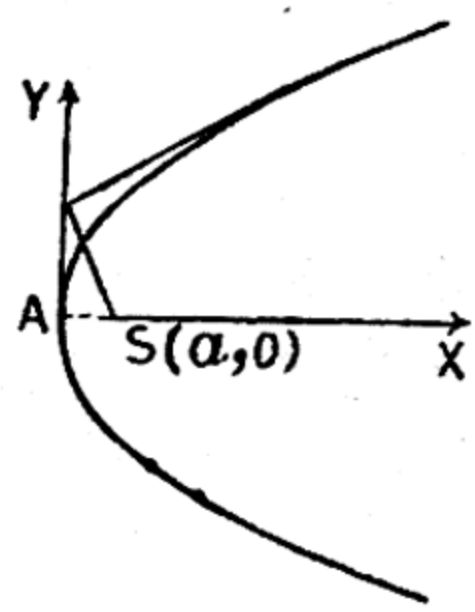
or $2a=x-x_1=\text{AG}-\text{AN}=\text{NG}$

$\therefore$ NG=2*a*(constant)=half the latus rectum.

117. (*c*) The locus of the foot of the perpendicular from the focus on any tangent to a parabola is the tangent at the vertex.*

Let the equation of the parabola be $y^2=4ax$. The equation of *any* tangent to the parabola, in the *m*-form, is

$$y=mx+\frac{a}{m} \quad \ldots(1)$$



The slope of any line $\perp$ to it $=-\frac{1}{m}$ [−ve reciprocal]

$\therefore$ the equation of the $\perp$ from S(*a*, 0) on the tangent (1) is

$$y-0=-\frac{1}{m}(x-a) \qquad [y-y_1=m(x-x_1)]$$

or $$y=-\frac{1}{m}x+\frac{a}{m} \quad \ldots(2)$$

[To find the locus of the foot of the $\perp$ (*i.e.*, pt. of intersection of (1) and (2)), *i.e.*, to eliminate *m* from (1) and (2).

(Misc. Ex. 23, Chap. III, Rule)]

Subtracting (2) from (1), $0=\left(m+\frac{1}{m}\right)x$ or $x=0$, which is the required locus. This is the *y*-axis, *i.e.*, the tangent at the vertex.

[Art. 99, Cor.]

[Note. For problems relating to a tangent to a parabola (or conic), and perpendicular to it, take the equation of any tangent in the m-form.]

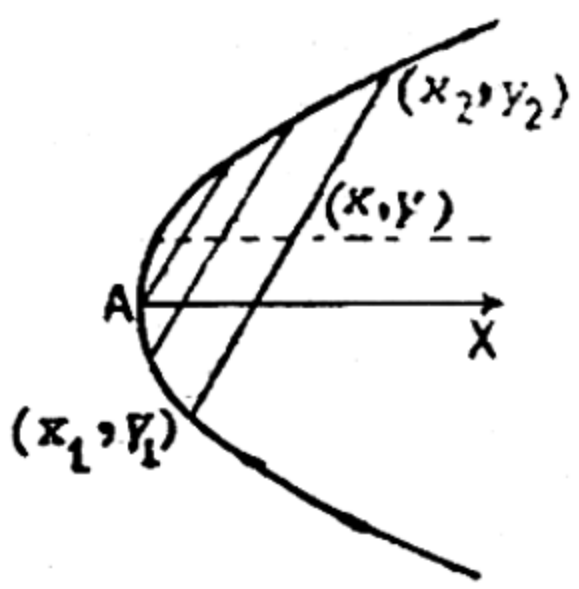
EXAMPLE

Find the locus of the foot of the perpendicular drawn from the vertex on a tangent to the parabola $y^2=4ax$. [*J. & K. U. 1955*]

117. (*d*) Diameter property. The locus of the middle points of a system of parallel chords of a parabola is a straight line parallel to the axis.†

Let the equation of the parabola be $y^2=4ax$.

(*i*) Let (x_1, y_1), (x_2, y_2) be the ends of *any* chord of the system and (x, y) its mid-pt.



Then $x=\frac{x_1+x_2}{2}$, $y=\frac{y_1+y_2}{2}$ ...(1)

Let *m* be the slope of the ‖ chords.

Then $$m=\frac{y_2-y_1}{x_2-x_1} \quad \ldots(2)$$

(*ii*) [To find the value of $\frac{y_2-y_1}{x_2-x_1}$.]

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*[*P. U. 1956S*] †[*P. U. 1954*]

$\because$ (x_1, y_1), (x_2, y_2) lie on the parabola,

$\therefore$ $y_1^2=4ax_1$ ...(3)

$y_2^2=4ax_2$ ...(4)

Subtracting (3) from (4),

$y_2^2-y_1^2=4a(x_2-x_1)$.

Factorising, $(y_2-y_1)(y_2+y_1)=4a(x_2-x_1)$

$\therefore$ $\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{4a}{y_2+y_1}$ ...(5)

(*iii*) Substituting the values of $\dfrac{y_2-y_1}{x_2-x_1}(=m)$, $y_2+y_1(=2y)$ from (2) and (1) in (5),

$$m=\frac{4a}{2y}$$ [Cancel 2]

or $$\mathbf{y=\frac{2a}{m}}$$, which is the required locus.

This is a st. line $\parallel$ to the x-axis, *i.e.*, $\parallel$ to the axis of the parabola.

[**N.B.** *The equation of the diameter (Def. Art. 84) of the parabola* $y^2=4ax$, *bisecting parallel chords of slope* m, *is* $\mathbf{y=\frac{2a}{m}}$.]

[Rule to find the locus of the middle points of a system of parallel chords of a parabola : *See Rule of Art. 84.*]

Vertex of a diameter. Def. *The* **vertex** *of a diameter of a parabola (or conic) is the point of intersection of the diameter and the parabola (or conic).*

EXAMPLE

**** Equation of a chord in terms of its mid-point. The equation of the chord of the parabola $y^2=4ax$, whose mid-point is (x_1, y_1), is**

$$\mathbf{yy_1-2ax=y_1^2-2ax_1}.$$

[**Hint.** Proceed as in Method I, solved Ex. 1, Art. 84. (Here the mid-pt. (x_1, y_1) lies on the diameter $y=\dfrac{2a}{m}$ (Art. 117, (*d*)).]

[****N.B.** *The equation of the chord of the parabola* $S=0$, *whose mid-point is* (x_1, y_1), *is* $\mathbf{T=S_1}$, *where* $T=0$ *is the equation of the tangent at* (x_1, y_1), *and* S_1 *is the result of substituting the co-ordinates of the point* (x_1, y_1) *in* S.]

117. (*e*) *The tangent at the vertex of a diameter of a parabola is parallel to the chords bisected by the diameter.**

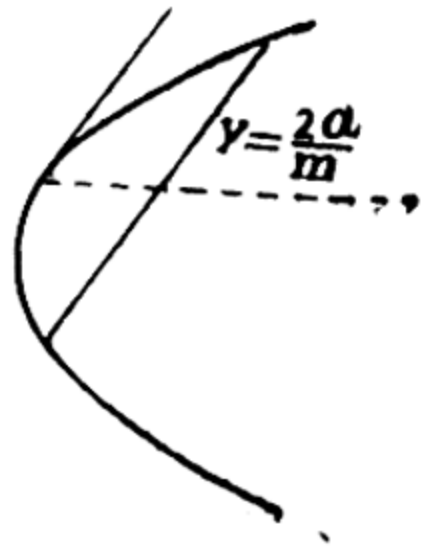
Method I.

Let the equation of the parabola be

$$y^2=4ax \quad \ldots (1)$$

(*i*) Let m be the slope of the $\parallel$ chords. Then the equation of the diameter bisecting them is

$$y=\frac{2a}{m} \quad \ldots (2) \quad [\text{ Art. 117, } (d) \text{ }]$$



(*ii*) It meets the parabola where, substituting the value of y from (2) in (1), $\dfrac{4a^2}{m^2}=4ax$,

*[*P. U. 1954*]

$\therefore \quad x=\frac{a}{m^2}$.

$\therefore$ the vertex of the diameter is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

(*iii*) The equation of the tangent at the vertex is

$$y\left(\frac{2a}{m}\right)=2a\left(x+\frac{a}{m^2}\right) \qquad [yy_1=2a(x+x_1)]$$

[Cancel $2a$]

or $$y=mx+\frac{a}{m},$$

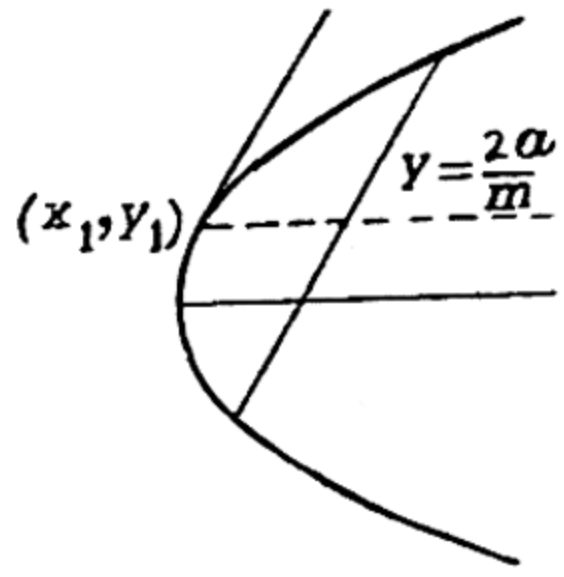
which is $\parallel$ to the chords. [$\because$ its slope$=m$]

Method II.

Let the equation of the parabola be $y^2=4ax$.

(*i*) Let m be the slope of the $\parallel$ chords.

Then the equation of the diameter bisecting them is $y=\frac{2a}{m}$ ...(1) [Art. 117,(*d*)]



Let (x_1, y_1) be the vertex of the diameter.

$\because$ (x_1, y_1) lies on (1),

$\therefore \quad y_1=\frac{2a}{m}$ ...(2)

(*ii*) The equation of the tangent at the vertex is

$$yy_1=2a(x+x_1).$$

(*iii*) $\therefore$ the slope of the tangent

$=\frac{2a}{y_1}$ [Substitute the value of y_1 from (2) (*Note this step*)]

$=\frac{2a}{\frac{2a}{m}}=m=$ the slope of the chords

$\therefore$ the tangent is $\parallel$ to the chords.

Note. The foregoing method is applicable to any *conic.*

[Art. 137,(*g*), Ex. 4]

MISCELLANEOUS EXAMPLES ON CHAPTER VI

1. Find the equation to the parabola with focus (a, b) and directrix $\frac{x}{a}+\frac{y}{b}=1$. [*P.U. 1944S*]

2. **Find the equation of the parabola whose focus is the origin, and whose directrix is the line $y-x=4$.**

Find also the length of the latus rectum, the equation of the axis and the co-ordinates of the vertex. [*P. U.*]

[Sol. (*ii*) *Length of the latus rectum* $=2$. [$\perp$ *from the focus* $(0, 0)$ *on the directrix* $x-y+4=0$].

$\therefore$ length of the latus rectum $=2.\ \frac{4}{\sqrt{2}}=4\sqrt{2}$.

(*iii*) *The axis is the line thro' the focus* $(0, 0)$ $\perp$ *to the directrix* $x-y+4=0$.

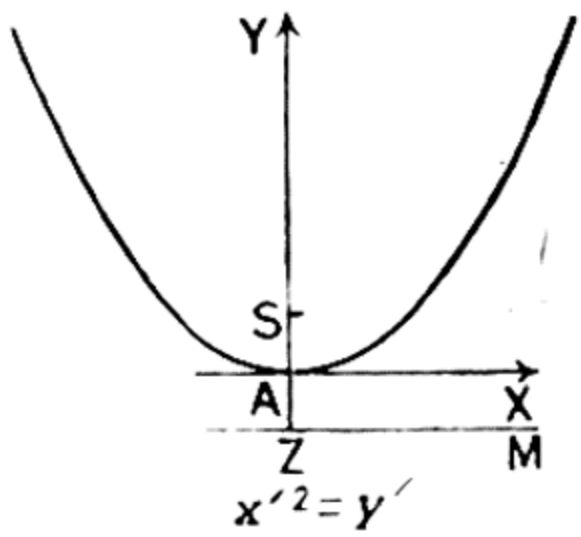
The equation of *any* line $\perp$ to $x-y+4=0$ is $x+y+k=0$ ...(1)
If it passes thro' (0, 0), then $0+0+k=0$, $\therefore$ $k=0$.
Substituting this value of k in (1), $x+y=0$, which is the equation of the axis.

(*iv*) *The vertex A is the mid-pt. of SZ, where Z is the pt. of intersection of the directrix* $x-y+4=0$ *... (1), and the axis* $x+y=0$ *... (2)*
From (1) and (2) [adding and subtracting], $2x+4=0$, $-2y+4=0$
$\therefore$ $x=-2$, $y=2$.
$\therefore$ Z is $(-2, 2)$. Also S is (0, 0).
$\therefore$ the vertex is $\left(\frac{0+(-2)}{2}, \frac{0+2}{2}\right)$ or $(-1, 1)$.]

3. Obtain the equation of a parabola whose focus is the point (1, 1) and whose directrix is the straight line x−y=3. Find also the length of the latus rectum, the equation of the axis and the co-ordinates of the vertex. [*Pesh. U. 1952*]

****4. Find the vertex, focus, latus rectum, axis and directrix of the parabola $x^2-y-2x=0$.** [*P. U. 1937S*]

[**Sol.** The equation of the parabola is
$$x^2-2x-y=0.$$
Transposing, $x^2-2x=y$.
Completing the square, $x^2-2x+1=y+1$
or $(x-1)^2=y+1$.
Put $x-1=x'$ and $y+1=y'$.
Then $x'^2=y'$, which is an upward parabola.
(Compare with $x^2=4ay$)
Here $4a=1$, $\therefore$ $a=\frac{1}{4}$.



(*i*) The vertex is $x'=0$, $y'=0$ or $x-1=0$, $y+1=0$ or $x=1$, $y=-1$.
$\therefore$ the vertex is $(1, -1)$.
(*ii*) The focus is $x'=0$, $y'=a$ or $x-1=0$, $y+1=\frac{1}{4}$ or $x=1$, $y=-\frac{3}{4}$.
$\therefore$ the focus is $(1, -\frac{3}{4})$.
(*iii*) The latus rectum $=4a=1$.
(*iv*) The axis is $x'=0$ or $x-1=0$.
(*v*) The directrix is $y'=-a$ or $y+1=-\frac{1}{4}$ or $4y+4=-1$ or $4y+5=0$.]

5. Find the co-ordinates of the vertex, the co-ordinates of the focus, the length of the latus rectum and the equation of the directrix of each of the following parabolas :
(*i*) $(y+3)^2=2(x+2)$. [*P.U. 1949S*]
(*ii*) $y^2=2x+2$. [*C. U.*]

****6. Find the vertex, focus and latus rectum of the parabola**
$$y^2=4y-4x.$$
[*C. U.*]
Find also the axis and the directrix. [*P(P). U.*]

****7. Find the vertex, axis, focus, and latus rectum of the parabola :**
(*i*) $4y^2+12x-12y+39=0$. [*J. & K. U.*]
(*ii*) $x^2-2ax+2ay=0$. [*B.H.U.*]

[**Hint.** (*i*) Divide thro'out by 4 (to make coeff. of square term=1).]

8. In the parabola $y^2=3x$, (*i*) find the equation to the chord through the vertex and the negative end of the latus rectum.

(*ii*) Find the equation to *any* chord through the point on the curve whose abscissa is 12.

9. Show that the tangent to the parabola $y^2-4ax=0$ at the point (x_1, y_1) is perpendicular to the tangent at the point $\left(\frac{a^2}{x_1}, -\frac{4a^2}{y_1}\right)$.

10. (*i*) **Interpret the equations $y^2=4ax$ and $x^2=4by$.** [*P. U.*]

(*ii*) **Find the co-ordinates of the common points of the loci represented by the equations.** [*P. U. 1943S*]

(*iii*) **Find the equation of their common chord.** [*P. U.*]

[**Sol.** (*ii*) The given equations are $y^2=4ax$ ...(1)

$x^2=4by$ ...(2)

Substituting the value of $y \left(= \frac{x^2}{4b} \right)$ from (2) in (1),

$$\frac{x^4}{16b^2}=4ax$$

$\therefore$ $x=0$ or, dividing by x, $x^3=64ab^2$ or $x=4a^{\frac{1}{3}}b^{\frac{2}{3}}$.

Substituting these values of x one by one in $y=\frac{x^2}{4b}$,

if $x=0$, $y=0$;

if $x=4a^{\frac{1}{3}}b^{\frac{2}{3}}$, $y=\frac{16a^{\frac{2}{3}}b^{\frac{4}{3}}}{4b}=4a^{\frac{2}{3}}b^{\frac{1}{3}}$.

$\therefore$ the pts. of intersection are (0, 0) and $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$.]

[**Caution.** When the values of x (for the points of intersection) have been found, substitute them one by one in the equation *from which the value of y in terms of x was found, i.e., in the equation* $y=\frac{x^2}{4b}$ (*and not in the equation* $y^2=4ax$).]

11. **Find the equation of the tangent to the parabola $x^2=4ay$ at the point (x_1, y_1).** [*P. U. 1944S*]

[**Hint.** Use the Rule of Art. 63.]

12. **Obtain the angle at which the parabolas $y^2=4ax$ and $x^2=4by$ intersect.** [*P(P). U. 1954*]

[**Hint.** Proceed as in Ex. 10, and find the pts. of intersection (0, 0) and $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$. The equations of the tangents to the parabolas at $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$ are $y(4a^{\frac{2}{3}}b^{\frac{1}{3}})=2a(x+4a^{\frac{1}{3}}b^{\frac{2}{3}})$, [$yy_1=2a(x+x_1)$]

$x(4a^{\frac{1}{3}}b^{\frac{2}{3}})=2b(y+4a^{\frac{2}{3}}b^{\frac{1}{3}})$. [$xx_1=2b(y+y_1)$ (Rule of Art. 64)]

The slopes of the tangents are

$$\frac{2a}{4a^{\frac{2}{3}}b^{\frac{1}{3}}}=\frac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}, \quad \frac{4a^{\frac{1}{3}}b^{\frac{2}{3}}}{2b}=\frac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}} ;$$

find the angle between the tangents.

Note. The equations of the tangents to the parabolas at (0, 0) are

$y(0)=2a(x+0)$, [$yy_1=2a(x+x_1)$]

$x(0)=2b(y+0)$ [$xx_1=2b(y+y_1)$ (Rule of Art. 64)]

or $x=0$(*i.e.*, y-axis), $y=0$(*i.e.*, x-axis), which are at rt. $\angle$s, $\therefore$ at the origin, the parabolas intersect at rt. $\angle$s.]

13. Find the slopes of the three normals to $y^2=4x$ through the point (9, 6) and write down their equations. [*P. U. 1934S*]

14. Show that the line $y=2x+1$ cuts $y^2=8x$ in coincident points. Show also that it cuts $5x^2+5y^2=1$ in coincident points.

15. Find the length of the chord $4y=3x+8$ intercepted by the parabola $y^2=8x$. [*P. U. 1956*]

16. The normal at P$(3, 2\sqrt{3})$ to the parabola $y^2=4x$ cuts the curve again at Q. Show that $PQ=\frac{32}{3}$. [*P. U. 1950S*]

[**Caution.** See 'Caution' in Ex. 10. When the values of x (for the points of intersection) have been found, substitute them one by one *in the equation of the normal and not in the equation of the parabola.*

The same caution should be observed when finding the points of intersection of a line and a curve. See step (*iii*) of the Rule given in Note 2, Art. 69.]

17. Prove that the straight line $3x+4y+6=0$ is a tangent to the parabola $2y^2=9x$, and find its point of contact. [*C. U.*]

18. Prove that the straight line $ty=x+at^3$ touches the parabola $y^2=4ax$ and find the co-ordinates of the point of contact. [*P.U. 1931*]

19. (*i*) Prove that the straight line $x+y=1$ touches the parabola $y=x-x^2$. [*A. U. 1945*]

****(*ii*) Show that the straight line $7x+6y=13$ is a tangent to the parabola $y^2-7x-8y+14=0$. Find the point of contact.** [*P.U. 1953S*]

20. Find the condition that $x\cos\alpha+y\sin\alpha=p$ may touch the parabola $y^2=4ax$, and obtain the co-ordinates of the point of contact, if the line touches the parabola.

21. Find the equation to the common tangent of the parabolas $y^2=4ax$ and $x^2=4by$. [*J. & K. U. 1952*]

[**Hint.** The equation of *any* tangent to the parabola $y^2=4ax$, in the m-form, is $y=mx+\frac{a}{m}$ ...(1)

The equation of the second parabola is $x^2=4by$ ...(2)

Substituting the value of y from (1) in (2),

$x^2=4b\left(mx+\frac{a}{m}\right)$ or $x^2-4bmx-\frac{4ab}{m}=0$ ...(3)

If (1) touches (2) also, the quadratic (3) has equal roots.

$\therefore\ 16b^2m^2+\frac{16ab}{m}=0,\quad \therefore\ m^3=-\frac{a}{b}$ or $m=-\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$.

Substitute this value of m in (1).]

22. A straight line touches the circle $x^2+y^2=2$, and also the parabola $y^2=8x$. Show that its equation is $y=\pm(x+2)$. [*D. U.*]

23. A tangent to the parabola $y^2=4x$ makes an angle of 45° with the line $y=2x+3$, find its equation and the point of contact. [*C.U.*]

24. Show that the point (4, 5) lies outside the parabola $y^2=4x$, and find the equations of the tangents from this point to the parabola. [*P. U.*]

25. Show that the angle θ between the two tangents to the parabola $y^2=4ax$ through the point (x_1, y_1) is given by

$$\tan\theta=\pm\frac{\sqrt{y_1^2-4ax_1}}{a+x_1}.$$ [*P. U. 1933S*]

[**Hint.** The equation of *any* tangent to the parabola, in the *m*-form, is $y=mx+\frac{a}{m}$. If it passes thro' (x_1, y_1), then $y_1=mx_1+\frac{a}{m}$

or $m^2x_1-my_1+a=0$ ...(1)

which is a quadratic in m. Let m_1, m_2 be the roots.

Now $\tan\theta=\pm\frac{m_1-m_2}{1+m_1m_2}=\pm\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{1+m_1m_2}$ ...(2)

But from (1), $m_1+m_2=\frac{y_1}{x_1}$, $m_1m_2=\frac{a}{x_1}$. Substitute in (2).]

26. Find the locus of the point of intersection of two tangents to the parabola $y^2=4ax$, which make an angle α with one another. [*B. U. 1930*]

27. Find the equation of the locus of the point of intersection of two tangents to a parabola which make an angle of 45° with one another. [*P. U. 1957*]

28. Find the co-ordinates of the point of intersection of the tangents $y=mx+\frac{a}{m}$, $y=m'x+\frac{a}{m'}$. Show that the locus of their intersection is a straight line when mm' is constant.

29. Find the angle at which the polars of (2, 1) with respect to the circle $x^2+y^2=1$ and the parabola $y^2=6x$ intersect.

30. Find the co-ordinates of the point of intersection of tangents drawn to the parabola $y^2=4ax$ at the points where it is cut by the straight line $x\cos\alpha+y\sin\alpha=p$. [*P. U. 1949S*]

31. If t_1, t_2 are the parameters of the ends of a focal chord of the parabola $y^2=4ax$, then $t_1t_2=-1$. [*P. U. 1957*]

[**Hint.** It will be found that the equation of the line joining the pts. t_1, t_2 on the parabola is $y(t_1+t_2)=2x+2at_1t_2$. (Art. 115)

If it passes thro' $(a, 0)$, then $0=2a+2at_1t_2$.]

Cor. Important. The ends of a focal chord are

$$\mathbf{(at^2, 2at)\ and\ \left(\frac{a}{t^2}, -\frac{2a}{t}\right).}$$

$$\left[\text{For, } t_2=-\frac{1}{t},\ \therefore\ (at_2^2, 2at_2) \text{ becomes } \left(\frac{a}{t^2}, -\frac{2a}{t}\right).\right]$$

32. (*i*) Show that the product of the abscissae of the extremities of a focal chord of a parabola is a constant. [*P. U. 1954S*]

(*ii*) Prove that the tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity. [*P. U.*]

33. Prove that the tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix. [*P. U. 1957*]

34. Prove that the locus of the middle points of focal chords of the parabola $y^2=4ax$ is $y^2=2a(x-a)$. [*P. U. 1946*]

35. If O is any point on the axis of a parabola and POQ be any chord passing through O, and if PM and QN be the ordinates of P and Q, prove that AM.AN$=$AO^2. Also prove that PM.QN$=-4$a.AO. [*P. U.*]

36. At any point P of a parabola a normal PG is drawn intersecting the axis in G. Prove that PG is a geometric mean between the latus rectum and SP. [*P. U. 1939*]

[**Sol.** The equation of the normal at P(x_1, y_1) is

$y-y_1=-\frac{y_1}{2a}(x-x_1)$...(1). It meets the x-axis where, putting $y=0$ in (1), $2a=x-x_1$ (dividing by $-y_1$) or $x=2a+x_1$. $\therefore$ G is $(2a+x_1, 0)$, $\therefore$ by distance formula, $PG^2=4a^2+y_1^2=4a^2+4ax_1=4a(a+x_1)=4a.SP$ (Art. 95-I, Cor.).]

37. The normal at the point $(at_1^2, 2at_1)$ meets the parabola again in $(at_2^2, 2at_2)$, prove that $t_2=-t_1-\frac{2}{t_1}$. [*P. U. 1954S*]

38. In the parabola $y^2=4ax$, find the equation to the locus of the middle point of the normal PG at P, where G is on the axis. [*A. U.*]

[**Hint.** Find the equation of the normal at P$(at^2, 2at)$. It will be found that G is $(2a+at^2, 0)$.

$\therefore$ the co-ordinates of the mid-pt. of PG are $x=a+at^2$, $y=at$. Eliminate t.]

****39.** Prove that the locus of the foot of the perpendicular from the focus on the normal to a parabola is a parabola. [*P. U. 1935*]

[**Hint.** To find the pt. of intersection of $y=mx-2am-am^3$ ...(1)

and $y-0=-\frac{1}{m}(x-a)$ ... (2)

Subtracting (2) from (1),

$0=\left(m+\frac{1}{m}\right)x-\left(\frac{a}{m}+2am+am^3\right)$ or $0=\frac{1+m^2}{m}x-\frac{a(1+m^2)^2}{m}$.

$\therefore$ $x=a(1+m^2)$ ...(3), $\therefore$ from (1), $y=-am$ ...(4). Eliminate m from (3) and (4).]

40. The general equation to a system of parallel chords in the parabola $y^2=6x$ is $3x-y+k=0$. What is the equation to the corresponding diameter ?

CHAPTER VII

[☞ **This chapter should be omitted by the Intermediate students of the Jammu & Kashmir University in 1959 (*and after*).]**

THE ELLIPSE

SECTION I

EQUATION OF AN ELLIPSE

118. Ellipse. Def. An *ellipse* is the locus of a point which moves* so that its distance from a fixed point is in a constant ratio, less than one, to its distance from a fixed straight line.

Defs. (*i*) The fixed point is called the **focus**, and is usually denoted by S. †

(*ii*) The fixed straight line is called the **directrix.**

(*iii*) The constant ratio is called the **eccentricity**, and is denoted by e (the initial letter of the word *eccentricity*).

***Note.** It is assumed that the point moves in the plane in which the fixed point and the fixed straight line lie.

[**N. B. Definition property.** *If P is any point on an ellipse whose focus is S and eccentricity e, and PM ⊥ on the directrix, then*

$$\frac{SP}{PM} = e \text{ or } \mathbf{SP = e.\ PM.} \quad (e < 1)\ ]$$

An instructive Note. For the parabola $e = 1$,

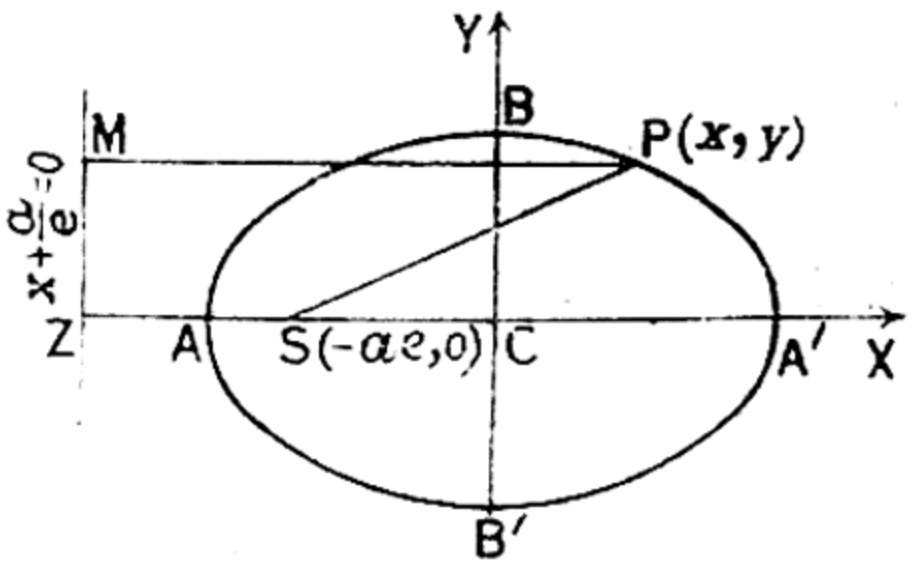
$\therefore$ SP $= e$. PM becomes SP = PM. [Cf. Art. 94, (*c*), N.B.]

119. (*a*) **Standard form. To find the equation of an ellipse in the form** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. ‡

Let S be the focus, ZM the directrix, and e the eccentricity of the ellipse. From S draw SZ ⊥ on ZM. Divide SZ internally and externally at A, A′ in the ratio $e : 1$, so that $\frac{SA}{AZ} = \frac{e}{1}$, $\frac{SA'}{A'Z} = \frac{e}{1}$,

or $SA = e.\ AZ$ ... (1)

$SA' = e.\ A'Z$ ... (2)§



†The initial letter of the word *Sun* because every planet describes an ellipse with the *Sun* in a focus.

‡ [*J. & K. U. 1954*]

§ In this and in what follows, only the *magnitudes* of the line-segments along AA′ have been considered.

[Then A, A′ lie on the ellipse (Def.).]

Bisect AA′ in C. Let $AA'=2a$, then $CA=CA'=a$.

(*i*) Adding (1) and (2),

$$SA+SA'=e\,(AZ+A'Z)$$

or $$AA'=e\,(CZ-CA+CZ+CA')$$

or $$2a=e.2CZ \qquad [\because\ CA=CA']$$

or $$CZ=\frac{a}{e}.$$

Subtracting (1) from (2),

$$SA'-SA=e(A'Z-AZ)$$

or $$CS+CA'-(CA-CS)=e.AA'$$

or $$2CS=e.2a \qquad [\because\ CA=CA']$$

or $$CS=ae.$$

(*ii*) Take C as origin, CA′ as the x-axis, and CY $\perp$ to it as the y-axis. Then the co-ordinates of S are $(-ae, 0)$, and the equation of ZM is $x=-\frac{a}{e}$ [(Art. 16) $\parallel$ to y-axis at a distance $\frac{a}{e}$ to the *left*]

or $x+\frac{a}{e}=0$.

(*iii*) Let P(x, y) be *any* pt. on the ellipse. Join SP, and from P draw PM $\perp$ on ZM.

Then $SP=e.PM$ ...(3) [Def. (Art. 118)]

But $SP=\sqrt{(x+ae)^2+(y-0)^2}=\sqrt{(x+ae)^2+y^2}$,

and $PM=\dfrac{x+\frac{a}{e}}{1}$ [$\perp$ from (x, y) on $x+\frac{a}{e}=0$]

$$=x+\frac{a}{e}.$$

$\therefore$ from (3), $\sqrt{(x+ae)^2+y^2}=e\left(x+\frac{a}{e}\right)=ex+a$

or, squaring, $x^2+2aex+a^2e^2+y^2=e^2x^2+2aex+a^2$

or $x^2(1-e^2)+y^2=a^2(1-e^2)$

or, dividing thro' out by $a^2(1-e^2)$*, $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$

or, putting $a^2(1-e^2)=b^2$, [$\because\ e<1$]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which is the required equation.

Cor. Important. $CS=ae$, $CZ=\frac{a}{e}$, $b^2=a^2(1-e^2)$.

Note 1. Geometrical meaning of b. The y-axis meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1) where, putting $x=0$ in (1), $\frac{y^2}{b^2}=1$ or $y^2=b^2$

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*Why this step. To get 1 on the R. H. S. as required in the equation to be found, *viz.*, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

or $y=\pm b$, *i.e.*, in two points B, B′ (Fig. of Art. 119, (*a*)) such that

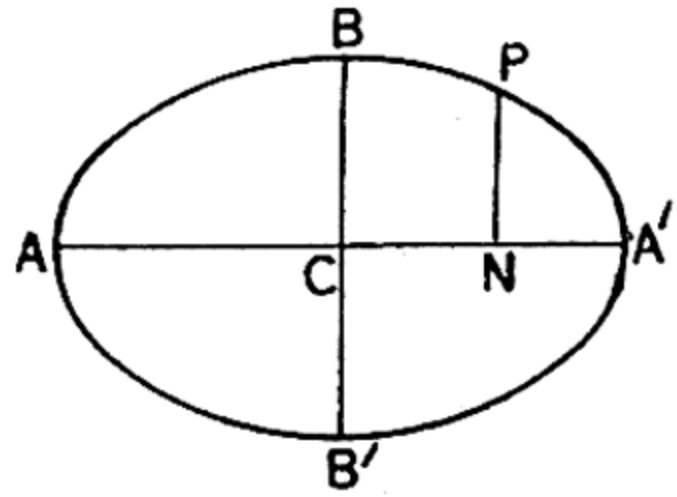
$$CB=CB'=b.$$

Defs. (*i*) The points A, A′ are called the **vertices** of the ellipse. (*ii*) AA′ is called the **major axis,*** (*iii*) BB′ is called the **minor axis**; and (*iv*) the two together are called the **axes** of the ellipse. (*v*) C is called the **centre**† of the ellipse.

Note 2. Standard form. Since $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the simplest form of the equation of an ellipse, it is called the **standard form.**

Note 3. To interpret geometrically the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.‡

From *any* pt. P on the ellipse draw PN ⊥ on the major axis, then



$$\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\therefore \frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}=\frac{(a+x)(a-x)}{a^2},$$

i.e., $$\frac{PN^2}{CB^2}=\frac{(CA+CN)(CA'-CN)}{CA^2}=\frac{AN.A'N}{CA^2}$$

or $\frac{PN^2}{AN.A'N}=\frac{CB^2}{CA^2}$, which is the required geometrical interpretation.

An instructive Note 4. A circle is a particular case of an ellipse.

For let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Putting $b=a$, it becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1$ or $x^2+y^2=a^2$, which is a circle.

[Aid to memory for standard results in the circle $x^2+y^2=a^2$. In the corresponding results for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, put $b^2=a^2$.]

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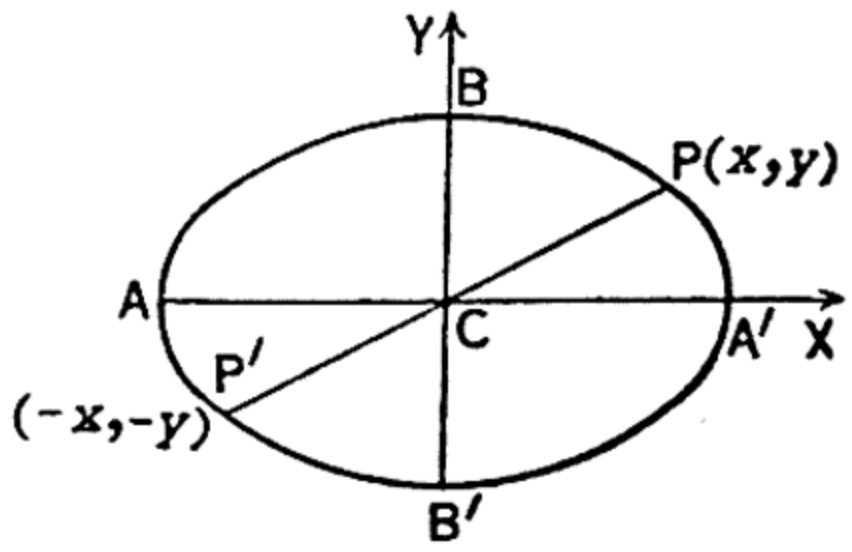
*So called, because AA′ is **greater** than BB′. For $b^2=a^2(1-e^2)=a^2-a^2e^2$

$\therefore b^2<a^2$ or $b<a$ or $2b<2a$,

or $2a>2b$, *i.e.*, AA′ > BB′.

†So called because C *bisects every chord of the ellipse, which passes through it.* For, let P(x, y) be *any* pt. on the ellipse, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Substituting the co-ordinates of the pt. P′$(-x, -y)$ in the equation of the ellipse, we get $\frac{(-x)^2}{a^2}+\frac{(-y)^2}{b^2}=1$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is true,

∴ P′ also lies on the ellipse.

Now the mid-pt. of PP′ is $\left[\frac{x+(-x)}{2}, \frac{y+(-y)}{2}\right]$ or (0, 0), *i.e.*, C.

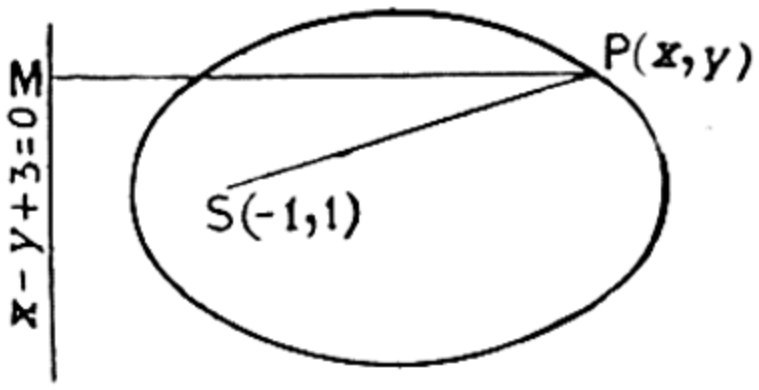
‡[*B. H. U. 1945*]

119. (*b*) Equation of an ellipse referred to any focus and directrix.

Example. Find in its most simplified form the equation of the ellipse whose focus is the point (−1,1), whose directrix is the straight line x−y+3=0 and whose eccentricity is ½. [*J. & K. U. 1957*]

Let S(−1, 1) be the focus.

(*i*) Let P(*x*, *y*) be *any* pt. on the ellipse. Join SP, and from P draw PM ⊥ on the directrix $x-y+3=0$.



(*ii*) Then $SP=e.PM$ ...(1)

(*iii*) But $SP=\sqrt{(x+1)^2+(y-1)^2}$, [Distance formula]

$$PM=\frac{x-y+3}{\sqrt{2}},$$ [⊥ distance formula]

and $e=\frac{1}{2}$,

$$\therefore \text{ from (1), } \sqrt{(x+1)^2+(y-1)^2}=\tfrac{1}{2}.\frac{x-y+3}{\sqrt{2}}$$

or, squaring, $x^2+2x+1+y^2-2y+1=\frac{1}{8}(x^2+y^2+9-2xy-6y+6x)$*

or $8(x^2+y^2+2x-2y+2)=x^2-2xy+y^2+6x-6y+9$

or $7x^2+2xy+7y^2+10x-10y+7=0$, which is the required equation.

EXAMPLES

1. (*i*) Find the equation of the ellipse whose centre is the origin, whose axes are the axes of co-ordinates, and whose major axis=6, minor axis=4. [*A. U. 1934*]

(*ii*) For what value of *p* does the ellipse $px^2+4y^2=1$ pass through the points (±1, 0). Find the lengths of its two axes. [*C. U. 1935*]

2. Find the equation to the ellipse referred to its axes as axes of co-ordinates

(*i*) whose major axis=8, and eccentricity=½.

(*ii*) which passes through the points (−3, 1) and (2, −2). Find also its eccentricity. [*C. U. 1945*]

3. Find the equation of the ellipse (referred to its axes as the axes of x and y respectively) which passes through the point (−3, 1) and has the eccentricity $\sqrt{\frac{2}{5}}$. [*C. U. 1948*]

4. Obtain the equation of the ellipse having the focus at (1, 2), directrix 3x+4y=5 and eccentricity=½. [*P. U.*]

120. Tracing an ellipse. *To trace the curve* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

(*i*) **Axes-intersections.** The *x*-axis meets the curve where, putting $y=0$ in (1), $\frac{x^2}{a^2}=1$ or $x^2=a^2$ or $x=\pm a$, *i.e.*, in the pts. A(−*a*, 0),

*How this step. See the footnote on p. 155.

A′$(a, 0)$. (See Fig. given below.)

Again, the y-axis meets the curve where, putting $x=0$ in (1), $\frac{y^2}{b^2}=1$ or $y^2=b^2$ or $y=\pm b$, *i.e.*, in the pts. B$(0, b)$, B′$(0, -b)$.

(*ii*) **Symmetry.** From (1), $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$

or $$y=\pm\frac{b}{a}\sqrt{a^2-x^2} \quad \ldots (2)$$

$\therefore$ for every value of x numerically $<a$, there are two equal and opposite values of y, $\therefore$ the curve is symmetrical about the x-axis.

Again from (1), $\frac{x^2}{a^2}=1-\frac{y^2}{b^2}=\frac{b^2-y^2}{b^2}$

or $$x=\pm\frac{a}{b}\sqrt{b^2-y^2} \quad \ldots(3)$$

$\therefore$ for every value of y numerically $<b$, there are two equal and opposite values of x, $\therefore$ the curve is symmetrical about the y-axis.

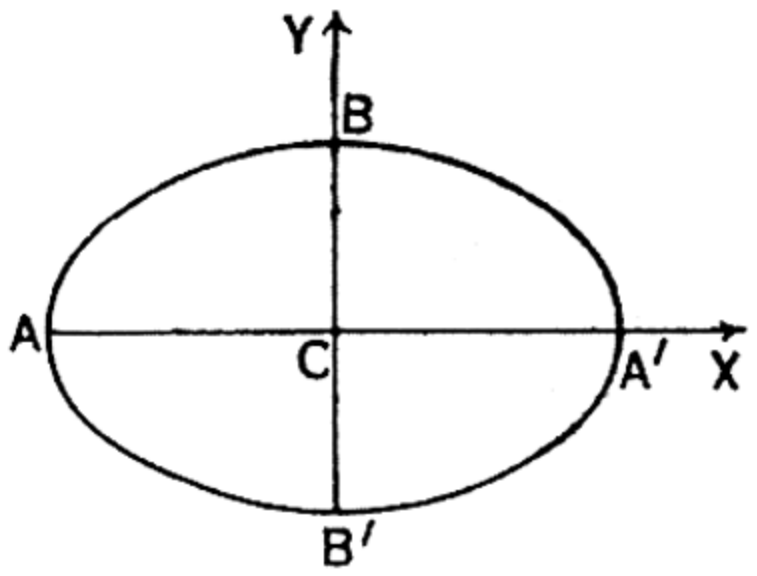
(*iii*) **Imaginary values.** From (2), y is imaginary when x is numerically $>a$, $\therefore$ no portion of the curve lies to the right of A′ or to the left of A.

Again from (3), x is imaginary when y is numerically $>b$, $\therefore$ no portion of the curve lies above B or below B′.

(*iv*) **Space occupied.** From (2), taking +ve sign with the square root, $y=\frac{b}{a}\sqrt{a^2-x^2}$, $\therefore$ as x (numerically $<a$) increases, y decreases.

$\therefore$ the curve is closed.

Hence the shape of the curve is that shown in the Fig.



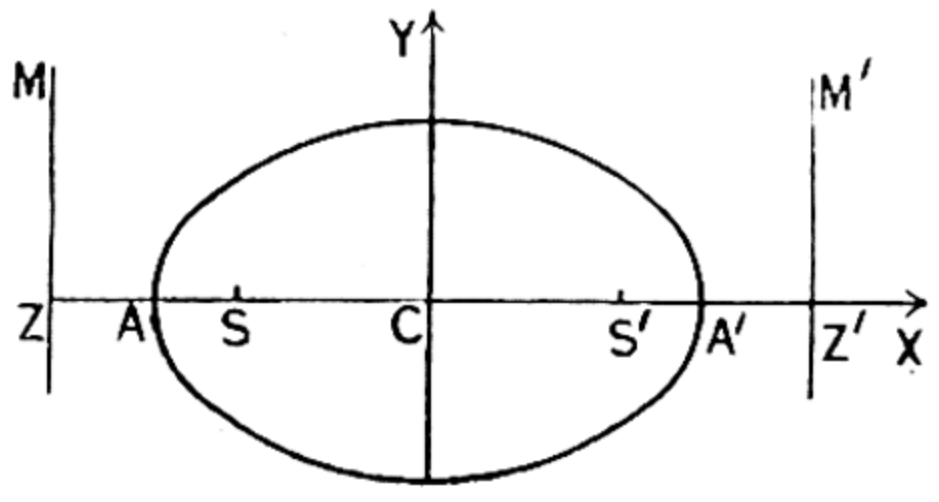
[**Rule to trace an ellipse whose equation is given :** *See Rule of Art. 96.*]

[**Aid to memory.** See Art. 96, **Aid to memory.**]

Cor. The ellipse has a second focus and a second directrix.

Proof. $\because$ the ellipse is symmetrical about the y-axis (Art. 120, (*ii*)),

$\therefore$ if on the x-axis the pts. S′, Z′ are taken such that



$CS'=CS$, $CZ'=CZ$, then the pt. S′ will also be a focus of the ellipse, and Z′M′, the line thro' Z′ $\perp$ to CZ′, will be the *corresponding* directrix.

[**Important.** *The foci S, S′ lie on the major axis, on opposite sides of C such that* $\mathbf{CS=CS'=ae}$; *the corresponding directrices ZM, Z′M′ are perpendicular to the major axis on opposite sides of C,*

such that $$\mathbf{CZ=CZ'=\frac{a}{e}}.$$

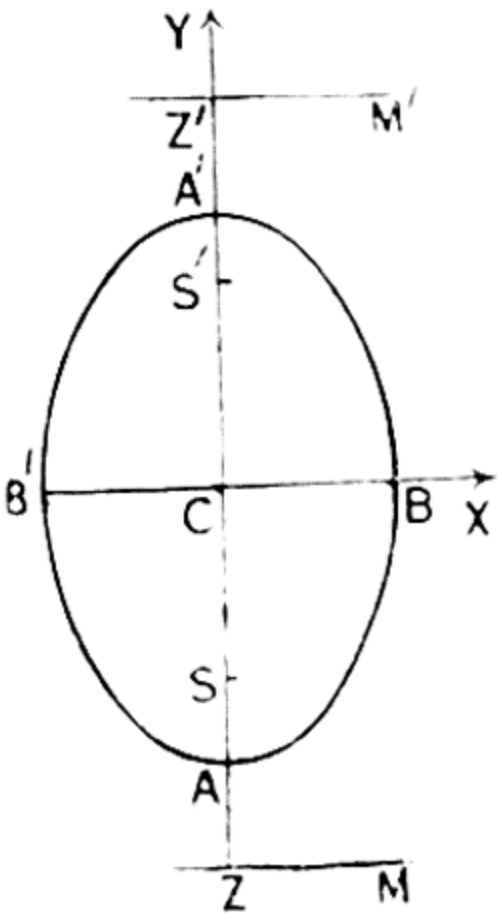
N.B. *The foci of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a^2>b^2)$ *are* $(\pm\ \mathbf{ae},\ 0)$, *and the equations of the corresponding directrices are* $\mathbf{x=\pm\frac{a}{e}}$.]

EXAMPLES

1. Trace the curve $\mathbf{\frac{x^2}{b^2}+\frac{y^2}{a^2}=1}$ $\mathbf{(a^2>b^2)}$.

[**Hint.** The equation of the curve is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ ... (1)

Proceed as in Art. 120. Here in step (*i*), *Axes-intersections*, the x-axis meets the curve where, putting $y=0$ in (1), $\frac{x^2}{b^2}=1$ or $x^2=b^2$ or $x=\pm b$, *i.e.*, in the pts. B$(b, 0)$, B′$(-b, 0)$.



Again, the y-axis meets the curve where, putting $x=0$ in (1), $\frac{y^2}{a^2}=1$ or $y^2=a^2$ or $y=\pm a$, *i.e.*, in the pts. A′$(0, a)$, A$(0, -a)$.

$\because$ $a>b$, $\therefore$ the major axis of the ellipse lies along the y-axis.]

Cor. To find the foci and the directrices of the ellipse $\mathbf{\frac{x^2}{b^2}+\frac{y^2}{a^2}=1}$. The foci lie on the *major* axis on opposite sides of C such that $CS=CS'=ae$.

$\therefore$ the foci are $(0, \pm ae)$.

Again, the directrices ZM, Z′M′ are $\perp$ to the major axis on opposite

sides of C such that $CZ=CZ'=\frac{a}{e}$,

$\therefore$ the equations of the corresponding directrices are $y=\pm\frac{a}{e}$.

Note. First and second standard forms. The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ...(1)$$

of an ellipse in which $a^2>b^2$, *i.e., denominator of* $x^2>$ *denominator of* y^2, is called the **first standard form,** and (changing x to y and y to x in (1)), $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ ...(2) in which $a^2>b^2$, *i.e., denominator of* $y^2>$ *denominator of* x^2, is called the **second standard form.**

Important. *To write down the standard results, the student should fix in his memory, the positions (Figs.) of the ellipses with respect to the co-ordinate axes.*

[**Aid to memory.** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in which denominator of $x^2>$ denominator of y^2, is an ellipse with its major axis horizontal.

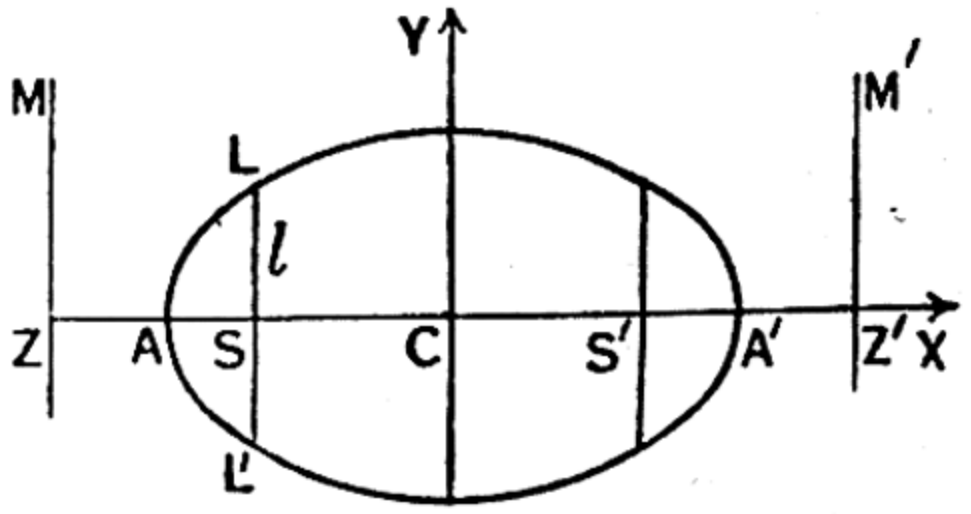
$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, in which denominator of $y^2>$ denominator of x^2, is an ellipse with its major axis vertical.]

2. Find the equation of the ellipse referred to its centre whose foci are the points (2, 0) and (−2, 0) and whose eccentricity is $\frac{1}{3}$.

[*P(P). U. 1951*]

Put down the equations of its directrices. [*P(P). U.*]

121. Length of the latus rectum. To find the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.*



The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let LSL′ be the latus rectum, and SL=l.
Then the pt. L($-ae$, l) lies on the ellipse

$\therefore \quad \frac{a^2e^2}{a^2}+\frac{l^2}{b^2}=1$ or $\frac{l^2}{b^2}=1-e^2$

*[*A. U. 1945*]

or $l^2=b^2(1-e^2)$ $\left[\text{ But } b^2=a^2(1-e^2), \quad \therefore\ 1-e^2=\frac{b^2}{a^2}\right]$

$=b^2.\frac{b^2}{a^2}=\frac{b^4}{a^2}$

$\therefore \quad l=\frac{b^{2*}}{a}$,

$\therefore$ LSL′ $=2l=2\ \mathbf{\frac{b^2}{a}}$.

[**N.B.** *The latus rectum of each of the ellipses* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ is $2\ \mathbf{\frac{b^2}{a}}$.

The result may be proved in the second case by drawing the Fig. (Art. 120, Ex. 1) and proceeding as in Art. 121. Here L(l, $-ae$) lies on the ellipse.]

Note. The plural of 'latus rectum' is *latera recta*.

Cor. 1. To find the equations of the latera recta. The equation of the latus rectum through S is $\mathbf{x=-ae}$ [(Art. 16) ∥ to y-axis at a distance *ae* to the *left*], and that of the latus rectum through the second focus S′ is $\mathbf{x=ae}$.

Cor. 2. To find the co-ordinates of the ends of the latera recta. The co-ordinates of L, the *positive* end of the latus rectum through S, are $\left(\mathbf{-ae}, \frac{\mathbf{b^2}}{\mathbf{a}}\right)$, and those of L′, the *negative* end, are $\left(\mathbf{-ae}, -\frac{\mathbf{b^2}}{\mathbf{a}}\right)$. Similarly the co-ordinates of the *positive* end of the latus rectum through S′ are $\left(\mathbf{ae}, \frac{\mathbf{b^2}}{\mathbf{a}}\right)$, and those of the *negative* end are $\left(\mathbf{ae}, -\frac{\mathbf{b^2}}{\mathbf{a}}\right)$.

Example. Find the eccentricity, co-ordinates of the foci, and the length of the latus rectum of the ellipse :

(1) $9x^2+16y^2=144$. [*P.U. 1945*]

(2) $25x^2+16y^2=400$; find also the directrices and equations of the latera recta. [*P(P). U. 1955*]

(1) The equation of the ellipse is $9x^2+16y^2=144$.

Dividing thro'out by 144, $\frac{x^2}{16}+\frac{y^2}{9}=1$.

[R.H.S. $=1$ and coeffs. of x^2, y^2 as denoms. (*Note this step*)]

Here (greater denominator) $a^2=16$, (lesser denominator) $b^2=9$.

(*i*) $b^2=a^2(1-e^2)$, $\therefore\ 9=16(1-e^2)=16-16e^2$,

$\therefore\ 16e^2=7$, $\therefore \quad e=\frac{\sqrt{7}}{4}$.

(*ii*) The foci are ($\pm ae$, 0), *i.e.*, $\left(\pm 4.\frac{\sqrt{7}}{4},\ 0\right)$ or $(\pm\sqrt{7},\ 0)$.

• ∵ l=SL which, from the Fig., is +ve.

(*iii*) The latus rectum $=2 \cdot \frac{b^2}{a}=2 \cdot \frac{9}{4}=\frac{9}{2}$.

Note. (*iv*) The equations of the *directrices* are $x=\pm\frac{a}{e}$,

i.e., $x=\pm\frac{4}{\sqrt{7/4}}=\pm\frac{16}{\sqrt{7}}$ or $\sqrt{7}x\pm16=0$.

(2) The equation of the ellipse is $25x^2+16y^2=400$.

Dividing thro'out by 400, $\frac{x^2}{16}+\frac{y^2}{25}=1$ ...(1)

[R.H.S.$=1$ and coeffs. of x^2, y^2 as denoms. (*Note this step*)]

Here denom. of $y^2>$ denom. of x^2 [*i.e.*, *second standard form*]

$\therefore$ the major axis lies along the **y-axis**.

From (1), (greater denominator) $a^2=25$,(lesser denominator) $b^2=16$.

(*i*) $b^2=a^2(1-e^2)$, $\therefore$ $16=25(1-e^2)=25-25e^2$, $\therefore$ $25e^2=9$, $\therefore$ $e=\frac{3}{5}$.

(*ii*) The foci lie on the *major* axis, *i.e.*, on the y-axis on opposite sides of C such that

$$CS=CS'=ae=5.\frac{3}{5}=3,$$

$\therefore$ the foci are $(0, \pm3)$.

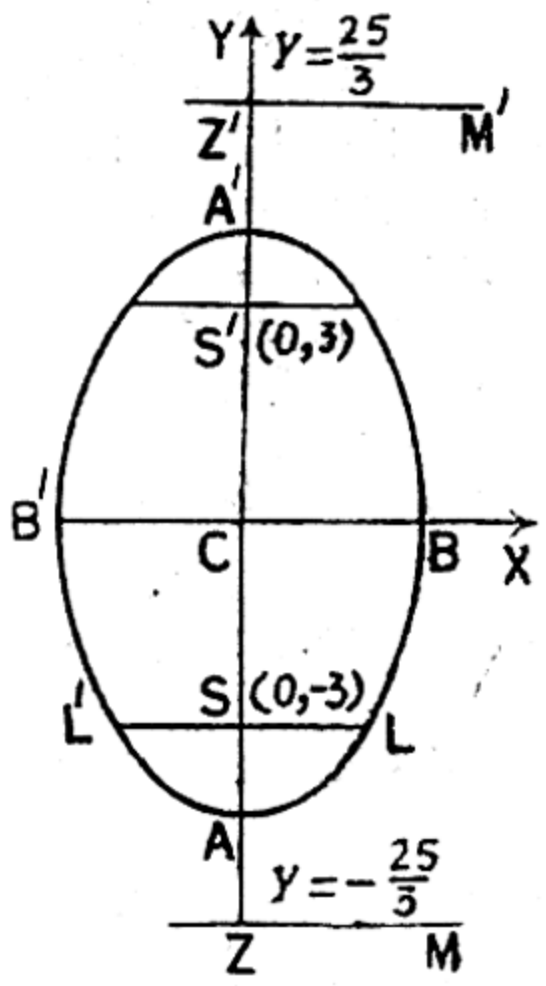
(*iii*) The latus rectum $=2.\frac{b^2}{a}=2.\frac{16}{5}=\frac{32}{5}$.

(*iv*) The directrices ZM, Z'M' are $\perp$ to the major axis, *i.e.*, $\perp$ to the y-axis or $\parallel$ to the x-axis on opposite sides of C such that

$$CZ=CZ'=\frac{a}{e}=\frac{5}{\frac{3}{5}}=\frac{25}{3},$$

$\therefore$ the equations of the directrices are

$$y=\pm\frac{25}{3} \text{ or } 3y\pm25=0.$$



(*v*) The latera recta are chords thro' the foci $\perp$ to the major axis, *i.e.*, $\perp$ to the y-axis or $\parallel$ to the x-axis.

$\therefore$ their equations are $y=\pm ae$, *i.e.*, $y=\pm5.\frac{3}{5}=\pm3$ or $y\pm3=0$.

EXAMPLES

1. Find the eccentricity of the ellipse whose latus rectum is 4 inches and the distance of the vertex from the nearest focus is 1·5 inches. [*C. U. 1944*]

2. Find the equation of the ellipse referred to its centre whose minor axis is equal to the distance between the foci and whose latus rectum is 2.

3. The latus rectum of an ellipse is 5 and its eccentricity $=\frac{2}{3}$, find its equation referred to the centre as origin of co-ordinates. [*B. H. U. 1946*]

4. (*a*) **Find the eccentricity, the co-ordinates of the foci, and the length of the latus rectum of the ellipse**

(*i*) $2x^2+3y^2-1=0$, (*ii*) $4x^2+3y^2=1$. [*P. U. 1956S*]

(*b*) Find the semi-major axis, semi-minor axis, the eccentricity, and the foci of the ellipse $4x^2+9y^2=25$. [*P(P). U. 1956S*]

5. Find the eccentricity, the distance between the foci and the latus rectum of the ellipse $3x^2+4y^2=12$. [*P. U.*]

6. Find the eccentricity of the ellipse

(*i*) **if its latus rectum be equal to one-half its minor axis.** [*P(P). U. 1951S*]

(*ii*) **if the latus rectum is half of its major axis.** [*A. U. 1948*]

7. Find the eccentricity of the ellipse if the distance between the foci is equal to the length of the latus rectum. [*P. U. 1946S*]

SECTION II

TANGENTS AND NORMALS

122. Equation of the tangent. To find the equation of the tangent at the point (x_1, y_1) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.*

[Method of chord.] The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let (x_2, y_2) be any other pt. on the ellipse.

(*i*) The equation of the line passing thro' (x_1, y_1), (x_2, y_2) is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad ...(1)$$

(*ii*) [To find the value of $\frac{y_2-y_1}{x_2-x_1}$.]

$\because$ (x_1, y_1), (x_2, y_2) lie on the ellipse,

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad ...(2)$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \quad ...(3)$$

Subtracting (2) from (3), $\frac{x_2^2-x_1^2}{a^2}+\frac{y_2^2-y_1^2}{b^2}=0$.

Factorising, $\frac{(x_2-x_1)(x_2+x_1)}{a^2}+\frac{(y_2-y_1)(y_2+y_1)}{b^2}=0$

or $$\frac{(y_2-y_1)(y_2+y_1)}{b^2}=-\frac{(x_2-x_1)(x_2+x_1)}{a^2}$$

$$\therefore \quad \frac{y_2-y_1}{x_2-x_1}=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}.$$

(*iii*) Substituting this value of $\frac{y_2-y_1}{x_2-x_1}$ in (1), *the equation of the chord thro' (x_1, y_1), (x_2, y_2) is*

$$y-y_1=-\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}(x-x_1) \quad ...(4)$$

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*[*P. U. 1951S*]

(*iv*) Putting $x_2=x_1$, $y_2=y_1$ in (4), the equation of the tangent at (x_1, y_1) is

$$y-y_1=-\frac{b^2}{a^2}\cdot\frac{2x_1}{2y_1}(x-x_1) \qquad \text{[Cancel 2]}$$

or, *multiplying both sides by* $\frac{y_1}{b^2}$*, (Note this step)*

$$(y-y_1)\frac{y_1}{b^2}=-\frac{x_1}{a^2}(x-x_1) \text{ or } \frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}$$

or $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

$$=1 \qquad \left[\because \frac{x_1^2}{a^2}+\frac{y_1^2}{b}=1, \text{ from (2)}\right]$$

$\therefore \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, which is the required equation.

[**N.B.** *The equation of the tangent at the point* (x_1, y_1) *of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.]

[**Rule to find the equation of the tangent at the point** (x_1, y_1) **of an ellipse (equation in the standard form) :** *See Rule of Art. 63.*]

[**Rule to write down the equation of the tangent at the point** (x_1, y_1) **of an ellipse (equation in the standard form) :** *See Rule of Art. 64.*]

EXAMPLES

1. Find the equation of the tangent to the ellipse $4x^2+3y^2=24$ at the point $(\sqrt{3}, 2)$. [*D. U. 1932*]

2. Show that (*i*) the tangents at the extremities of the major axis of an ellipse are parallel to the minor axis, and (*ii*) the tangents at the extremities of the minor axis are parallel to the major axis.

3. Find the equations to the tangents at the ends of the latera recta of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and show that they pass through the intersections of the axis and the directrices. [*P. U. 1942S*]

[**Sol.** (*ii*) The equation of the tangent at $L\left(-ae, \frac{b^2}{a}\right)$ is

$$\frac{x(-ae)}{a^2}+\frac{y\left(\frac{b^2}{a}\right)}{b^2}=1 \qquad \left[\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1\right]$$

or $$-\frac{ex}{a}+\frac{y}{a}=1 \text{ or } -ex+y=a \quad ...(1)$$

Substituting the co-ordinates of $Z\left(-\frac{a}{e}, 0\right)$ in (1), we get

$$-e\left(-\frac{a}{e}\right)+0=a$$

***Why this step.** To take all the x's and a^2 on one side, and all the y's and b^2 on the other.

or $a=a$, which is true, $\therefore$ the tangent at L passes thro' Z. Similarly the tangent at L′ passes thro' Z, and the tangents at the ends of the latus rectum thro' S′ pass thro' Z′.]

123. Equation of the normal. To find the equation of the normal at the point (x_1, y_1) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.*

The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(*i*) The equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \qquad \text{[Art. 122]}$$

$$\therefore \text{ the slope of the tangent}=-\frac{\frac{x_1}{a^2}}{\frac{y_1}{b^2}}=-\frac{b^2x_1}{a^2y_1}.$$

(*ii*) $\therefore$ the slope of the normal $=\frac{a^2y_1}{b^2x_1}$. [−ve reciprocal]

(*iii*) $\therefore$ the equation of the normal at (x_1, y_1) is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1) \qquad [y-y_1=m(x-x_1)]$$

or, *multiplying both sides by* $\frac{b^2}{y_1}$†, *(Note this step)*

$$(y-y_1)\frac{b^2}{y_1}=\frac{a^2}{x_1}(x-x_1) \text{ or } \frac{b^2y}{y_1}-b^2=\frac{a^2x}{x_1}-a^2$$

or $$\frac{\mathbf{a^2x}}{\mathbf{x_1}}-\frac{\mathbf{b^2y}}{\mathbf{y_1}}=\mathbf{a^2-b^2}.$$

[**Aid to memory.** Remember the L.H.S., *viz.*, $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}$. In this L.H.S., substitute the co-ordinates of the point (x_1, y_1). This is the R.H.S. of the required equation.]

Example. Obtain the equations of the tangent and normal to the ellipse $x^2+5y^2=14$ at (3, 1). [*P. U.*]

The equation of the ellipse is $x^2+5y^2=14$.

(*a*) The equation of the tangent at (3, 1) is [Art. 64, Rule]

[Here $x_1=3$, $y_1=1$]

$$x(3)+5y(1)=14 \text{ or } 3x+5y-14=0.$$

(*b*) **Method I.**

(*i*) The equation of the tangent at (3, 1) is $3x+5y-14=0$.

$\therefore$ the slope of the tangent $=-\frac{3}{5}$.

(*ii*) $\therefore$ the slope of the normal $=\frac{5}{3}$. [−ve reciprocal]

---

*[*P. U. 1955*]

†**Why this step.** To take all the x's and a^2 on one side, and all the y's and b^2 on the other. See the footnote on p. 194.

(*iii*) ∴ the equation of the normal at (3, 1) is

$$y-1=\frac{5}{3}(x-3) \qquad [y-y_1=m(x-x_1)]$$

or $3y-3=5x-15$ or $5x-3y-12=0$.

Method II. (Without finding the equation of the tangent.)

The equation of the ellipse is $x^2+5y^2=14$.

Dividing thro' out by 14, $\frac{x^2}{14}+\frac{5y^2}{14}=1$ or $\frac{x^2}{14}+\frac{y^2}{\frac{14}{5}}=1$

$$\left[\text{Form } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right]$$

$$\left[\text{Compare with } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right]$$

Here $a^2=14$, $b^2=\frac{14}{5}$,

∴ the equation of the normal at (3, 1) is [Here $x_1=3$, $y_1=1$]

$$\frac{14x}{3}-\frac{\frac{14}{5}y}{1}=14-\frac{14}{5} \qquad \left[\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2\right]$$

$$=\frac{56}{5} \qquad [\text{Cancel } 14]$$

or $\frac{x}{3}-\frac{y}{5}=\frac{4}{5}$ or $5x-3y=12$ or $5x-3y-12=0$.

EXAMPLES

1. Find the equations of the tangent and normal at the point of the ellipse $x^2+4y^2=25$, whose ordinate is 2.

2. Find the equations of the tangent and normal of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the end of the latus rectum (first quadrant). [*P.U.*]

3. Find (*i*) the equations of the tangents, and (*ii*) the equations of the normals at the ends of the latera recta of the ellipse $2x^2+3y^2=6$. [*P.U. 1944*]

SECTION III

AN ELLIPSE AND A LINE

124. Points of intersection. To find the points of intersection of the line $y=mx+c$ with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the line is $y=mx+c$ ... (1)

" " " ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (2)

(*i*) Substituting the value of y from (1) in (2),

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1 \quad \text{or} \quad \frac{x^2}{a^2}+\frac{m^2x^2+2mcx+c^2}{b^2}=1$$

or $b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2=a^2b^2$

or $x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0,$

which is a quadratic in x giving two values of x.

(*ii*) Substituting these values of x one by one in (1), we get the corresponding values of y.

These corresponding values of x and y are the co-ordinates of the required pts. of intersection.

[**Rule to find the points of intersection of a line and an ellipse:** *See Rule of Art. 69.*]

EXAMPLE

What are the co-ordinates of the points where $5x=4y$ meets the curve $\frac{x^2}{16}+\frac{y^2}{25}=1$? [*P. U.*]

125. Length of the intercept. To find the length of the intercept made by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on the line $y=mx+c$.*

The equation of the line is $y=mx+c$ ... (1)

,, ,, ,, ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (2)

(*i*) Substituting the value of y from (1) in (2),

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1 \text{ or } \frac{x^2}{a^2}+\frac{m^2x^2+2mcx+c^2}{b^2}=1$$

or $b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2=a^2b^2$

or $x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0$ ... (3)

which is a quadratic in x. Let x_1, x_2 be the roots.

[To find the value of x_2-x_1.]

From (3), $x_1+x_2=-\frac{2a^2mc}{a^2m^2+b^2}$, [Sum of the roots]

and $x_1x_2=\frac{a^2(c^2-b^2)}{a^2m^2+b^2}$. [Product of the roots]

$$\therefore \quad x_2-x_1=\sqrt{(x_2+x_1)^2-4x_2x_1}=\sqrt{\frac{4a^4m^2c^2}{(a^2m^2+b^2)^2}-\frac{4a^2(c^2-b^2)}{a^2m^2+b^2}}$$

$$=\sqrt{\frac{4a^4m^2c^2-4a^2(c^2-b^2)(a^2m^2+b^2)}{(a^2m^2+b^2)^2}}$$

$$=\frac{2a}{a^2m^2+b^2}\sqrt{a^2m^2c^2-(c^2-b^2)(a^2m^2+b^2)}$$

$$=\frac{2a}{a^2m^2+b^2}\sqrt{a^2m^2c^2-(a^2m^2c^2+b^2c^2-a^2m^2b^2-b^4)}$$

$$=\frac{2a}{a^2m^2+b^2}\sqrt{-b^2c^2+a^2m^2b^2+b^4}$$

$$=\frac{2ab}{a^2m^2+b^2}\sqrt{a^2m^2+b^2-c^2} \quad ...(4)$$

*[*P. U. 1955S*]

(*ii*) [To find the value of y_2-y_1.]

$\because$ (x_1, y_1), (x_2, y_2) lie on the *line*, [*Note this step*]

$\therefore$ $y_1=mx_1+c$ ... (5)

$y_2=mx_2+c$ ... (6)

Subtracting (5) from (6), $y_2-y_1=m(x_2-x_1)$ ...(7)

(*iii*) Now length of the intercept $=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(x_2-x_1)^2+m^2(x_2-x_1)^2}$ [From (7)]

$=(x_2-x_1)\sqrt{1+m^2}$

$=\dfrac{2ab}{a^2m^2+b^2}\sqrt{a^2m^2+b^2-c^2}.\ \sqrt{1+m^2}.$ [From (4)]

[Rule to find the length of the intercept made by an ellipse on the line $y=mx+c$: *See Rule of Art. 70.*]

Cor. Condition of tangency. To deduce the condition that the line may touch the ellipse.*

If the line touches the ellipse, the length of the intercept $=0$,

i.e., $\dfrac{2ab}{a^2m^2+b^2}\sqrt{a^2m^2+b^2-c^2}.\sqrt{1+m^2}=0$, $\therefore$ $a^2m^2+b^2-c^2$† $=0$

or $c^2=a^2m^2+b^2$, $\therefore$ $c=\pm\sqrt{a^2m^2+b^2}$, which is the required condition.

EXAMPLE

Show that the length of the line $y=x+2$ intercepted by the ellipse $3x^2+4y^2-16=0$ is $\frac{16}{7}\sqrt{2}$.

126. Condition of tangency. To find the condition that the line $y=mx+c$ may touch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.‡

[Method of intersections.]

The equation of the line is $y=mx+c$ ...(1)

,, ,, ,, ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ...(2)

(*i*) Substituting the value of y from (1) in (2),

$\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1$ or $\dfrac{x^2}{a^2}+\dfrac{m^2x^2+2mcx+c^2}{b^2}=1$

or $b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2=a^2b^2$

or $x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0$ ...(3)

which is a quadratic in x.

(*ii*) If the line *touches* the ellipse, the quadratic (3) has equal roots

$\therefore$ $4a^4m^2c^2-4(a^2m^2+b^2)a^2(c^2-b^2)=0$ $[b^2-4ac=0]$

or, *dividing by* $4a^2$, $a^2m^2c^2-(a^2m^2c^2-a^2m^2b^2+b^2c^2-b^4)=0$

or, $a^2m^2b^2-b^2c^2+b^4=0$

or, *dividing by* b^2, $a^2m^2-c^2+b^2=0$ or $c^2=a^2m^2+b^2$,

$\therefore$ $\mathbf{c=\pm\sqrt{a^2m^2+b^2}}$, which is the required condition.

---

*[*P.U. 1955S*] † *i.e.*, the factor involving c. ‡[*P(P). U. 1956*]

[Rule to find the condition that the line $y=mx+c$ may touch an ellipse : *See Rule of Art. 71.*]

[**Aid to memory.** Putting $b=a$, the ellipse becomes the circle $x^2+y^2=a^2$ (Art. 119, (*a*), Note 4), and the condition of tangency becomes

$c=\pm\sqrt{a^2m^2+a^2}$ or $c=\pm a\sqrt{1+m^2}$. (Cf. Art. 71.)]

Cor. 1. To find the equation of a tangent in terms of its slope. (See Art. 126.) Substituting the value of $c(=\pm\sqrt{a^2m^2+b^2})$ in (1), the equation of the tangent is $\mathbf{y=mx\pm\sqrt{a^2m^2+b^2}}$.

[**N.B.** *If m is the slope of a tangent to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *the equation of the tangent is* $\mathbf{y=mx\pm\sqrt{a^2m^2+b^2}}$.]

Cor. 2. Any tangent in the m-form. *The equation of any tangent to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *in the m-form, is* $\mathbf{y=mx+\sqrt{a^2m^2+b^2}}$.

[Taking +ve sign with the square root in the equation of Cor. 1]

[**Aid to memory.** Putting $b=a$, the ellipse becomes the circle $x^2+y^2=a^2$ (Art. 119, (*a*), Note 4). Hence

(*i*) *If m is the slope of a tangent to the circle $x^2+y^2=a^2$, the equation of the tangent is* $\mathbf{y=mx\pm a\sqrt{1+m^2}}$.

(*ii*) *The equation of any tangent to the circle $x^2+y^2=a^2$, in the m-form, is* $\mathbf{y=mx+a\sqrt{1+m^2}}$. (Cf. Art. 71, Cors. 1 and 2.)]

Example. Find the equations of the tangents to the ellipse $\frac{x^2}{16}+y^2=1$ which make with the axis of x an angle of 60°. [*D.U. 1942*]

The equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{1}=1$.

[Compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$]

Here $a^2=16$, $b^2=1$; $m=\tan 60°=\sqrt{3}$,
∴ the equations of the tangents are

$y=\sqrt{3}x\pm\sqrt{16(\sqrt{3})^2+1}$ [$y=mx\pm\sqrt{a^2m^2+b^2}$]
$=\sqrt{3}x\pm 7$

or $\sqrt{3}x-y+7=0$, $\sqrt{3}x-y-7=0$.

EXAMPLES

1. Prove that $y=mx+c$ cuts $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points. Also find the condition when the line is tangent to the ellipse. [*P.U. 1943*]

Deduce the corresponding condition for the circle $x^2+y^2=r^2$. [*B. U. 1944*]

[**Hint.** Proceed as in Ex. 1, Art. 71.]

2. Show that the line $y=x+\sqrt{\frac{5}{6}}$ touches the ellipse $2x^2+3y^2=1$; find the co-ordinates of the point of contact. [*C.U. 1955*]

3. Show that the straight line $y=mx+\sqrt{a^2m^2+b^2}$ touches

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **for all values of m.** [*P. U.*]

4. Determine c so that y=x+c may touch the ellipse

$$2x^2+3y^2=1.$$ [*P. U.*]

[**Sol. The equation of the ellipse is $2x^2+3y^2=1$ or $\frac{x^2}{\frac{1}{2}}+\frac{y^2}{\frac{1}{3}}=1$.

[Compare with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$]

Here $a^2=\frac{1}{2}$, $b^2=\frac{1}{3}$.

The equation of the line is $y=x+c$. [Compare with $y=mx+c$]

Here $m=1$, "c"$=c$.

If the line touches the ellipse, then

$c=\pm\sqrt{\frac{1}{2}(1)^2+\frac{1}{3}}$ [$c=\pm\sqrt{a^2m^2+b^2}$]

$=\pm\sqrt{\frac{5}{6}}$.]

5. Find the condition that the straight line x cos α+y sin α=p may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [*Bar. U. 1953*]

6. Condition of tangency. Find the condition that the straight line lx+my+n=0 may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [*P.U. 1957S*]

Find the co-ordinates of the point of contact. [*B. H. U. 1948*]

7. Find the equations of the tangents to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which make equal intercepts on the axes. [*P. U. 1952*]

8. Find the equations to the tangents to the ellipse $x^2+3y^2=3$

(*i*) which are parallel to the straight line y=4x+8.

(*ii*) which are perpendicular to the straight line x=4y+5.

[*P. U. 1956*]

SECTION IV

AN ELLIPSE AND A POINT

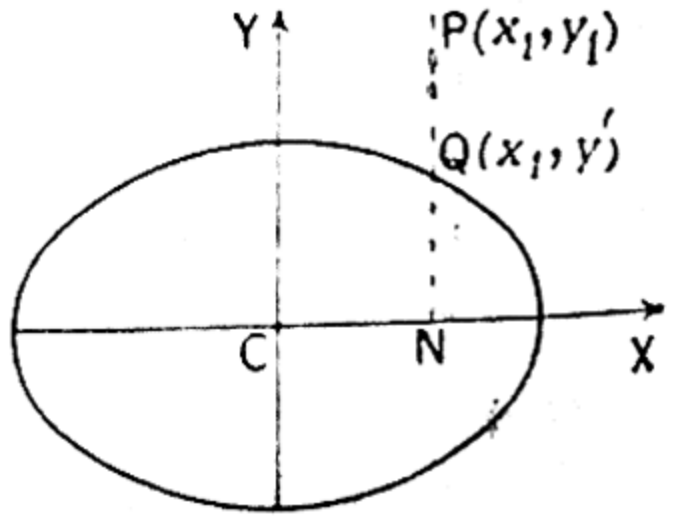
127. Test for a point to lie outside, on, or inside an ellipse.

To show that the quantity $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ *is positive, zero, or negative according as the point* (x_1, y_1) *lies outside, on, or inside the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Let P be the pt. (x_1, y_1).

From P draw PN $\perp$ on the x-axis to meet the ellipse in Q (x_1, y').

$\because$ Q lies on the ellipse

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y'^2}{b^2}=1$$

or $$\frac{x_1^2}{a^2}+\frac{y'^2}{b^2}-1=0 \qquad \ldots(1)$$ [R.H.S. zero]

Now $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1-0$$

[Substitute for 0 from (1)]

$$=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1-\left(\frac{x_1^2}{a^2}+\frac{y'^2}{b^2}-1\right)$$

$$=\frac{1}{b^2}\,[y_1^2-y'^2]=\frac{1}{b^2}\,[NP^2-NQ^2] \quad \ldots(2)$$

(*i*) If P lies outside the ellipse (as in the Fig.), NP > NQ,

$\therefore$ from (2)*, $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ is +ve.

(*ii*) If P lies on the ellipse, NP=NQ,

$\therefore$ from (2), $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ is zero.

(*iii*) If P lies inside the ellipse, NP < NQ,

$\therefore$ from (2), $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ is −ve.

[Rule to find whether a point lies outside, on, or inside an ellipse (equation in the standard form) : *See Rule of Art. 72.*]

EXAMPLES

1. Find whether the point (4, 5) lies within the curve $\frac{x^2}{16}+\frac{y^2}{25}=1$ or not. [*P. U.*]

2. Find if the point (4, −3) lies within or without the ellipse $5x^2+7y^2=140$. [*A. U. 1938*]

128. Two tangents from a point. To show that from any point two tangents can be drawn to an ellipse, and they are real, coincident, or imaginary according as the point is outside, on, or inside the ellipse.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and let the pt. be (x_1, y_1).

(*i*) The equation of *any* tangent to the ellipse, in the *m*-form, is

$$y=mx+\sqrt{a^2m^2+b^2}.$$ [Art. 126, Cor. 2]

(*ii*) If it passes thro' (x_1, y_1), then $y_1=mx_1+\sqrt{a^2m^2+b^2}$

or, transposing, $y_1-mx_1=\sqrt{a^2m^2+b^2}$

or, squaring, $y_1^2-2mx_1y_1+m^2x_1^2=a^2m^2+b^2$

---

* $\because$ R.H.S. is +ve, $\therefore$ L.H.S. is also +ve.

or $m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-b^2)=0$ ...(1)
which is a quadratic in m giving two values of m.
$\therefore$ from (x_1, y_1) two tangents can be drawn to the ellipse.

(*iii*) The tangents are real, coincident, or imaginary according as the roots of the quadratic (1) are real, equal, or imaginary,
i.e., according as $4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-b^2)$ is +ve, 0, or −ve
[b^2-4ac is +ve, 0, or −ve]
[Cancel 4]
or, according as $x_1^2y_1^2-(x_1^2y_1^2-b^2x_1^2-a^2y_1^2+a^2b^2)$ is +ve, 0, or −ve,
i.e., according as $b^2x_1^2+a^2y_1^2-a^2b^2$ is +ve, 0, or −ve
or, dividing thro'out by a^2b^2, [which is +ve]
according as $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ is +ve, 0, or −ve,
i.e., according as the pt. (x_1, y_1) lies outside, on, or inside the ellipse.
[Art. 127]

EXAMPLES

1. Find the equations of each of the two tangents to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}-1=0$ from the point $(-15, -7)$. [*C. U.*]

[Rule to find the equations of the tangents drawn from a given point to an ellipse : *See Rule of Method I, solved Ex., Art. 73.*]

2. Find the equations of the tangents drawn from the point (4, 1) to the ellipse $x^2+2y^2=6$, and prove that the angle between them is $\tan^{-1}\frac{3}{2}$.

3. Show that the tangent at the point $(-1, 1)$ to the ellipse $x^2+2y^2=3$ passes through the point (1, 2) and find the equation of the other tangent from (1, 2). [*P. U. 1950S*]

129. Director circle. To find the locus of the point of intersection of two perpendicular tangents to an ellipse.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(*i*) The equation of *any* tangent to the ellipse, in the m-form, is $y=mx+\sqrt{a^2m^2+b^2}$. [Art. 126, Cor. 2]

(*ii*) If it passes thro' (x_1, y_1), then $y_1=mx_1+\sqrt{a^2m^2+b^2}$
or, transposing, $y_1-mx_1=\sqrt{a^2m^2+b^2}$
or, squaring, $y_1^2-2mx_1y_1+m^2x_1^2=a^2m^2+b^2$
or $m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-b^2)=0$ ...(1)
which is a quadratic in m. Let m_1, m_2 be the roots.

(*iii*) If the tangents are $\perp$, $m_1m_2=-1$.

$\therefore$ from (1), $\frac{y_1^2-b^2}{x_1^2-a^2}=-1$ or $y_1^2-b^2=-x_1^2+a^2$
or $x_1^2+y_1^2=a^2+b^2$
$\therefore$ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y) (Art. 74)]
$x^2+y^2=a^2+b^2$, which is a circle.

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* [*P. U. 1952S*]

Note. This circle is called the **director circle** of the ellipse.

[Rule to find the locus of the point of intersection of two perpendicular tangents to an ellipse (equation in the standard form) : *See Rule of Art. 75.*]

EXAMPLES

1. (*a*) Define the director circle of an ellipse.

(*b*) Find the equation of the director circle of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad [\ P(P).\ U.\ 1955\ ]$$

2. Tangents to an ellipse make angles θ_1 and θ_2 with the major axis. Find the locus of their point of intersection if

(*i*) $\tan\theta_1+\tan\theta_2=k$; (*ii*) $\tan\theta_1\tan\theta_2=k$;

(*iii*) $\cot\theta_1+\cot\theta_2=k^2$;

(*iv*) $\theta_1+\theta_2$ is constant ($=2\alpha$). [*D. U. 1938*]

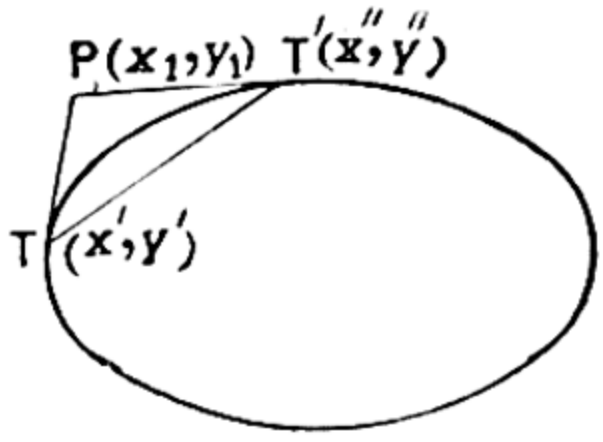
Chord of contact

130. Equation of the chord of contact. To find the equation of the chord of contact of tangents from (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.*$$

The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Let P be the pt. (x_1, y_1), and $T(x', y')$, $T'(x'', y'')$ the pts. of contact of the tangents from P to the ellipse.



(*i*) Then the equations of the tangents at T, T′ are

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1 \quad ...(1)$$

$$\frac{xx''}{a^2}+\frac{yy''}{b^2}=1 \quad ...(2)$$

(*ii*) ∵ both these tangents pass thro' P,

$$\therefore\ \frac{x_1x'}{a^2}+\frac{y_1y'}{b^2}=1 \quad ...(3)$$

$$\frac{x_1x''}{a^2}+\frac{y_1y''}{b^2}=1 \quad ...(4)$$

(*iii*) ∴ from (3) and (4), $T(x', y')$, $T'(x'', y'')$ lie on the line $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$†, *i.e.*, $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, which is, therefore, the required equation of the chord of contact. [Def. (Art. 76)]

[N.B. *The equation of the chord of contact of tangents from* (x_1, y_1) *to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.]

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* [*P. U. 1956S*]

†**How to write this equation.** In the equations (3) and (4) change (x', y') and (x'', y'') *to* (x, y).

EXAMPLES

1. Tangents are drawn from the point (1, 2) to the ellipse $3x^2+4y^2=5$. Find the equation to their chord of contact.

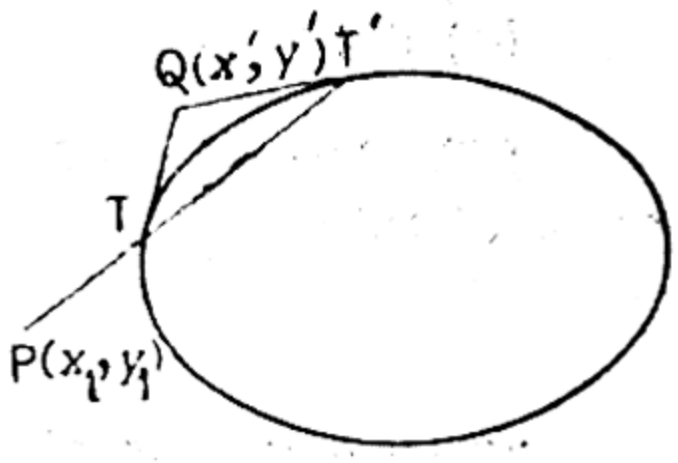
2. If the chord of contact of tangents from (α, β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=c^2$, show that the point (α, β) lies on the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$. [*P. U. 1956S*]

Pole and polar

131. Equation of the polar. To find the equation of the polar of the point (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Let P be the pt. (x_1, y_1), and PTT′ *any* line thro' P to meet the ellipse in T, T′. Let the tangents at T, T′ meet in Q(x', y').

(*i*) The equation of TT′, the chord of contact of tangents from Q, is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1. \qquad [\text{ Art. } 130\ ]$$

(*ii*) ∵ it passes thro' P,

$$\therefore \quad \frac{x_1x'}{a^2}+\frac{y_1y'}{b^2}=1.$$

(*iii*) ∴ the locus of Q(x', y') is [changing (x', y') to (x, y)]

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1, \text{ i.e., } \mathbf{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1},$$

which is the required equation of the polar of P [Def. (Art. 78)], and is clearly a st. line.

[**N.B.** *The equation of the polar of* (x_1, y_1) *w. r. t. the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* $\mathbf{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$.]

[**Aid to memory.** See Art. 79, **Aid to memory.**]

EXAMPLE

Show that the polar of the focus w.r.t. an ellipse is the corresponding directrix.

[Sol. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Then the equation of the polar of the focus $(-ae, 0)$ is $\frac{x(-ae)}{a^2}+\frac{y(0)}{b^2}=1$ $\left[\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1\right]$

or $-\frac{ex}{a}=1$ or $x=-\frac{a}{e}$, which is the corresponding directrix.]

132. Pole of a given line. To find the pole of the line $Ax+By+C=0$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.*

The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let (x_1, y_1) be the required pole.

(*i*) The equation of the polar of (x_1, y_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \text{ or } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0 \quad \ldots(1)$$

(*ii*) $\because$ it is the same as the equation of the given line

$$Ax+By+C=0 \quad \ldots(2)$$

(*iii*) $\therefore$ comparing coefficients in (1) and (2), $\frac{a^2}{A}=\frac{b^2}{B}=\frac{-1}{C}$

or $\frac{x_1}{Aa^2}=\frac{y_1}{Bb^2}=-\frac{1}{C}$, $\therefore\ x_1=-\frac{A}{C}a^2,\ y_1=-\frac{B}{C}b^2$

$\therefore$ the required pole is $\left(-\frac{A}{C}a^2, -\frac{B}{C}b^2\right)$.

[Rule to find the pole of a given line with respect to an ellipse : *See Rule of Art. 80.*]

EXAMPLES

1. Find with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ the pole of the line $y=mx+c$. [*P(P). U.*]

2. Find the pole of the line $x-2y+3=0$ with respect to the ellipse $3x^2+4y^2=12$.

3. Show that the focus of an ellipse is the pole of the corresponding directrix. [*P. U. 1945S*]

Properties of pole and polar

133. Reciprocal property. *If the polar of a point P with respect to an ellipse passes through a point Q, then the polar of Q passes through P.*

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P, Q be the pts. (x_1, y_1), (x_2, y_2). Then the equation of the polar of P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

If it passes thro' Q, then $\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}=1 \quad \ldots(1)$

Now the equation of the polar of Q is $\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \quad \ldots(2)$

Substituting the co-ordinates of P in (2), we get $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{a^2}=1$, which is true from (1). $\therefore$ the polar of Q passes thro' P.

*[*J. & K. U. 1951*]

EXAMPLE

If the polars of two points P and Q with respect to an ellipse meet in R, then R is the pole of the line PQ.

Use of parameter

134. Auxiliary circle. Def. The circle described on the major axis, AA′, of an ellipse as diameter, is called the **auxiliary circle** of the ellipse.

Cor. The **equation of the auxiliary circle** is $\mathbf{x^2+y^2=a^2}$.

For, its centre is C(0, 0), and its radius $CA=a$.

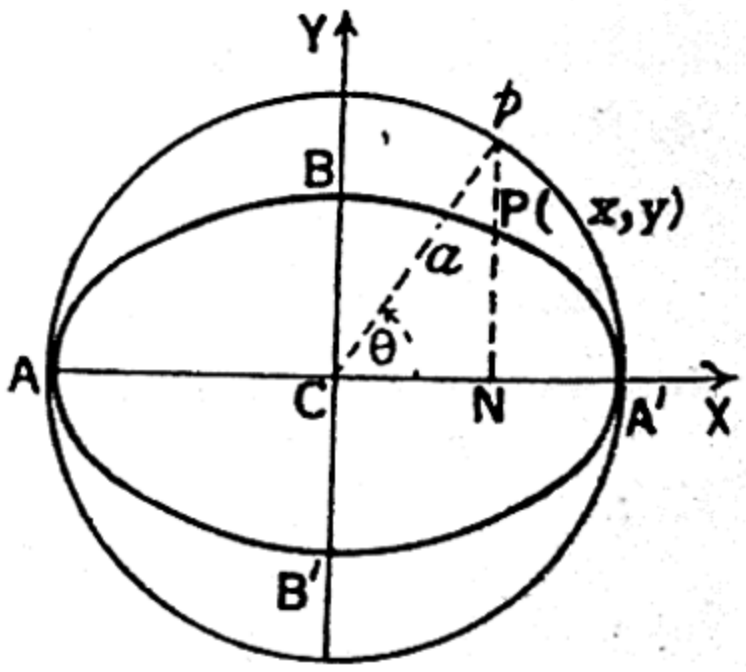
Eccentric angle. Def. [See Fig. of Art. 135.] Let P be *any* point on an ellipse, and NP its ordinate. Produce NP to meet the auxiliary circle in p, and join Cp. Then $\angle A'Cp$ is called the **eccentric angle** of P, and is generally denoted by θ.

****Note.** The point p on the auxiliary circle is said to *correspond* to the point P on the ellipse.

135. *To express the co-ordinates of any point on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *in terms of one variable* (called the **parameter**).*

The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Let P(x, y) be *any* pt. on the ellipse, and NP its ordinate.

Produce NP to meet the auxiliary circle in p, and join Cp.

Let $\angle A'CP=\theta$.

Then $x=CN$

[But $\frac{CN}{Cp}=\cos\theta$, $\therefore$ $CN=Cp\cos\theta$]

$=a\cos\theta$. [$\because$ $Cp=CA'=a$]

Substituting this value of x in the equation of the ellipse,

$\frac{a^2\cos^2\theta}{a^2}+\frac{y^2}{b^2}=1$, or $\frac{y^2}{b^2}=1-\cos^2\theta=\sin^2\theta$, or $y^2=b^2\sin^2\theta$

$\therefore$ $y=b\sin\theta$.†

$\therefore$ $\mathbf{x=a\cos\theta}$, $\mathbf{y=b\sin\theta}$, which are the required co-ordinates.

****Another method.** The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots(1)$$

Also $$\cos^2\theta+\sin^2\theta=1 \quad \ldots(2)$$

$\therefore$ comparing (1) and (2), $\frac{x^2}{a^2}=\cos^2\theta$, $\frac{y^2}{b^2}=\sin^2\theta$

or $$\frac{x}{a}=\cos\theta, \quad \frac{y}{b}=\sin\theta$$

---

*[*P.U. 1954*]

† $\because$ y=NP (in the Fig), which is +ve.

$\therefore$ **x=a cos θ, y=b sin θ,** which are the required co-ordinates.

[**Comparison of the two methods.** The foregoing method does not give the geometrical meaning of θ, *i.e.*, $\angle A'Cp=\theta$.]

Cor. Any point on the ellipse. *Any point on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* (**a cos θ, b sin θ**).

Note. Since the position of the point P (a cos θ, b sin θ) is known when the value of θ is given, hence the

Def. *The point whose co-ordinates are* (a *cos* θ, b *sin* θ) *is, for shortness, called the* "**point θ.**"

[**Aid to memory.** Putting $b=a$, the ellipse becomes the circle $x^2+y^2=a^2$ (Art. 119, (*a*), Note 4). Hence *any point on the circle* $x^2+y^2=a^2$ *is* (**a cos θ, b sin θ**). (Cf. Art. 82.)]

EXAMPLES

1. What are the parametric co-ordinates ?
Write down the parametric co-ordinates of
(*i*) $x^2+y^2=a^2$, (*ii*) $y^2=4ax$,
(*iii*) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [*P(P).U. 1949S*]

2. If θ is the eccentric angle of a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^3}{b^2}=1$, find the co-ordinates of P. [*B. U. 1943*]

Prove that the area of the triangle formed by three points on an ellipse, whose eccentric angles are α, β, γ, is

$$2ab \sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}.$$ [Cf. Art. 9, Ex. 2, (*b*), (*i*).]

3. **A system of chords of an ellipse is drawn from one end of its major axis, find the locus of the middle points.** [*J. & K. U. 1952*]

4. Find the locus of the middle points of chords of an ellipse which are drawn through the positive end of the minor axis.

****5.** Tangents are drawn from points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the circle $x^2+y^2=r^2$. Show that the chords of contact touch the ellipse $a^2x^2+b^2y^2=r^4$. [*A. U. 1937*]

136. To find the equation of the straight line joining two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose eccentric angles are given.*

The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let θ_1, θ_2 be the eccentric angles of the pts.

Then their co-ordinates are (a cos θ_1, b sin θ_1), (a cos θ_2, b sin θ_2). [Art. 135]

$\therefore$ the equation of the line passing thro' them is

*[*P. U. 1950S*]

$$y-b\sin\theta_1=\frac{b\sin\theta_2-b\sin\theta_1}{a\cos\theta_2-a\cos\theta_1}(x-a\cos\theta_1)$$

$$\left[\,y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\,\right]$$

$$=\frac{2b\cos\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}}{2a\sin\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}}(x-a\cos\theta_1)$$

[Cancel $2\sin\frac{\theta_1-\theta_2}{2}$ from the num. and denom.]

or, *dividing by b*,* (*Note this step*)

$$\text{or}\qquad \frac{y}{b}-\sin\theta_1=-\frac{\cos\frac{\theta_1+\theta_2}{2}}{\sin\frac{\theta_1+\theta_2}{2}}\left(\frac{x}{a}-\cos\theta_1\right)$$

$$\text{or}\qquad \frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}-\sin\theta_1\sin\frac{\theta_1+\theta_2}{2}$$

$$=-\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\cos\theta_1\cos\frac{\theta_1+\theta_2}{2}$$

$$\text{or}\qquad \frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}$$

$$=\cos\theta_1\cos\frac{\theta_1+\theta_2}{2}+\sin\theta_1\sin\frac{\theta_1+\theta_2}{2}$$

$$=\cos\left(\theta_1-\frac{\theta_1+\theta_2}{2}\right)=\cos\frac{\theta_1-\theta_2}{2}$$

$$\therefore\quad \frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}\quad\ldots(1)$$

which is the required equation.

Cor. *To deduce the equation of the tangent at the point* θ_1.†

Putting $\theta_2=\theta_1$ in (1), the equation of the tangent at the pt. θ_1 is $\frac{x}{a}\cos\theta_1+\frac{y}{b}\sin\theta_1=\cos 0=1$

$$\therefore\quad \frac{x}{a}\cos\theta_1+\frac{y}{b}\sin\theta_1=1,$$ which is the required equation.

EXAMPLES

1. Show that the point (a cos θ, b sin θ) lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of θ. What is the geometrical significance of θ? [*Bar. U. 1953*]

Obtain the equation of the tangent at the point. [*P.U. 1954*]

***Why this step.** To take x and a on one side, and y and b on the other:

†[*P. U. 1950*]

CHAPTER VIII

[☞This chapter should be omitted by the Intermediate students of the Jammu & Kashmir University.]

THE HYPERBOLA

SECTION I

EQUATION OF A HYPERBOLA

138. Hyperbola. Def.* A *hyperbola* is the locus of a point which moves† so that its distance from a fixed point is in a constant ratio, greater than one, to its distance from a fixed straight line.

Defs. (*i*) The fixed point is called the **focus,** and is usually denoted by S.

(*ii*) The fixed straight line is called the **directrix.**

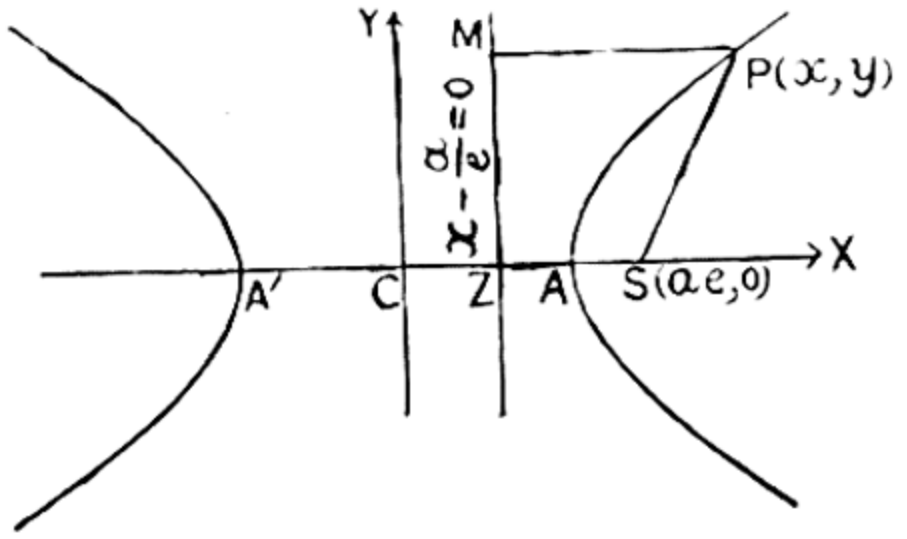
(*iii*) The constant ratio is called the **eccentricity,** and is denoted by e (the initial letter of the word *eccentricity*).

†**Note.** It is assumed that the point moves in the plane in which the fixed point and the fixed straight line lie.

[N.B. Definition property. *If P is any point on a hyperbola whose focus is S and eccentricity e, and $PM \perp$ on the directrix, then*

$$\frac{SP}{PM} = e \text{ or } \mathbf{SP = e.PM.}\ (e>1)]$$

139. Standard form. To find the equation of a hyperbola in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. ‡



Let S be the focus, ZM the directrix, and e the eccentricity of the hyperbola. From S draw SZ $\perp$ on ZM. Divide SZ internally and externally at A, A′ in the ratio e : 1, so that

$$\frac{SA}{AZ} = \frac{e}{1}, \quad \frac{SA'}{A'Z} = \frac{e}{1}\ \S$$

or
$$SA = e.AZ \quad ...(1)$$
$$SA' = e.A'Z \quad ...(2)$$

---

*[*P. U. 1953*] ‡[*P. U. 1955*]

§ ∵ $e>1$, $SA'>A'Z$, ∴ A′ lies on SZ produced.

[Then A, A′ lie on the hyperbola (Def.).]
Bisect AA′ in C. Let AA′=2*a*, then CA=CA′=*a*.
(*i*) *Adding* (*1*) *and* (*2*),

$$SA+SA'=e(AZ+A'Z)$$

or $CS-CA+CS+CA'=e.AA'$

or $2CS=e.2a$ [∵ CA=CA′]

or $CS=ae.$

Subtracting (*1*) *from* (*2*),

$$SA'-SA=e(A'Z-AZ)$$

or $AA'=e[CZ+CA'-(CA-CZ)]$

or $2a=e.2CZ$ [∵ CA=CA′]

or $CZ=\frac{a}{e}$.

(*ii*) Take C as origin, CA as the *x*-axis, and CY ⊥ to it as the *y*-axis. Then the co-ordinates of S are (*ae*, 0), and the equation of ZM is $x=\frac{a}{e}$ [(Art. 16) ‖ to *y*-axis at a distance $\frac{a}{e}$ to the right]

or $x-\frac{a}{e}=0.$

(*iii*) Let P(x, y) be *any* pt. on the hyperbola. Join SP, and from P draw PM ⊥ on ZM.

Then $SP=e.PM$ ...(3) [Def. (Art. 138)]

But $SP=\sqrt{(x-ae)^2+(y-0)^2}=\sqrt{(x-ae)^2+y^2}$,

and $PM=\frac{x-\frac{a}{e}}{1}$ [⊥ from (x, y) on $x-\frac{a}{e}=0$]

$=x-\frac{a}{e}$.

∴ from (3), $\sqrt{(x-ae)^2+y^2}=e\left(x-\frac{a}{e}\right)=ex-a$

or, squaring, $x^2-2aex+a^2e^2+y^2=e^2x^2-2aex+a^2$

or $x^2(e^2-1)^*-y^2=a^2(e^2-1)$ [∵ $e>1$]

or, dividing thro' out by $a^2(e^2-1)$ †, $\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$

or, putting $a^2(e^2-1)=b^2$, [∵ $e>1$]

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$ which is the required equation.

Cor. Important. $CS=ae$, $CZ=\frac{a}{e}$, $b^2=a^2(e^2-1)$.

---

*__Why this step.__ ∵ $e>1$, ∴ we have written the equation so that the coefficient of x^2 is (*positive*, *i.e.*,) e^2-1 and not $1-e^2$. Compare page 185, line 10 from the bottom, where, e being <1, we have written the equation so that the coefficient of x^2 is (positive, *i.e.*,) $1-e^2$.

†__Why this step.__ To get 1 on the R.H.S. as required in the equation to be found, *viz.*, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

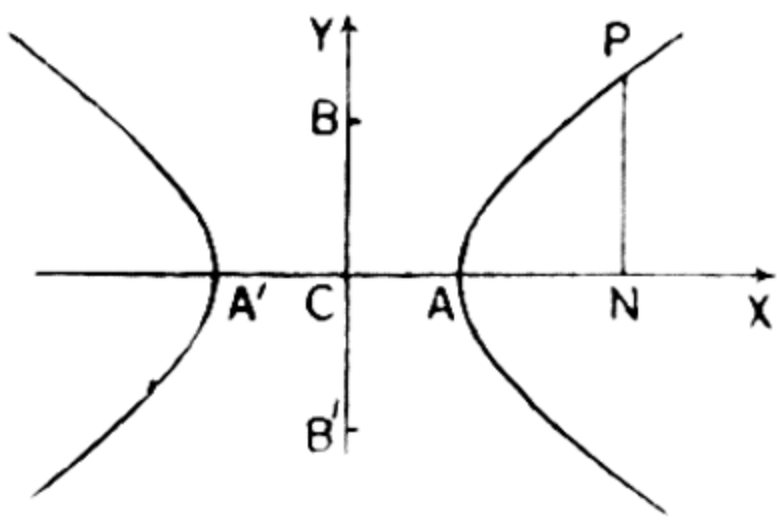
Note 1. Intersections of the hyperbola with the y-axis. The y-axis meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1) where, putting $x=0$ in (1), $-\frac{y^2}{b^2}=1$, or $y^2=-b^2$ or $y=\pm ib$ (imaginary), $\therefore$ the y-axis does not meet the hyperbola (in real pts.).

Defs. (*i*) The points A, A′ are called the **vertices** of the hyperbola. (*ii*) AA′ is called the **transverse* axis.** (*iii*) (See Fig. of Note 3 below.) If we take two points B, B′ on the y-axis such that $CB=CB'=b$, then BB′ is called the **conjugate axis.** (*iv*) C is called the **centre**† of the hyperbola.

Note 2. Standard form. Since $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is the simplest form of the equation of a hyperbola, it is called the **standard form.**

Note 3. To interpret geometrically the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

From *any* pt. P on the hyperbola draw PN $\perp$ on the transverse axis,



then $\because \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$\therefore \quad \frac{y^2}{b^2}=\frac{x^2}{a^2}-1=\frac{x^2-a^2}{a^2}$$

$$=\frac{(x-a)(x+a)}{a^2},$$

i.e.,
$$\frac{PN^2}{CB^2}=\frac{(CN-CA)(CN+CA')}{CA^2}=\frac{AN.A'N}{CA^2}$$

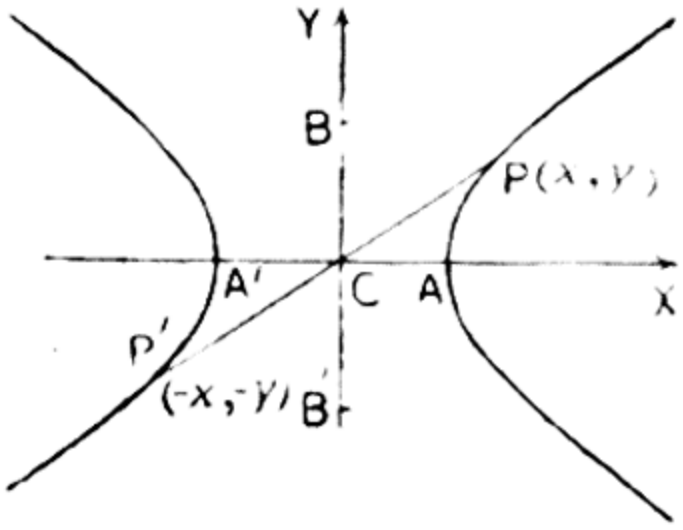
or $\frac{PN^2}{AN.A'N}=\frac{CB^2}{CA^2}$, which is the required geometrical interpretation.

Note 3. Rectangular hyperbola. Def. When the two axes of a hyperbola are equal in length (*i.e.*, $b=a$), it is called a **rectangular hyperbola.**

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*So called because it *lies across* (*i.e.*, cuts) the hyperbola (in A, A′).
[*trans* across.]

†So called because C *bisects every chord of the hyperbola, which passes through it.*
For, let $P(x, y)$ be *any* pt. on the hyperbola,



then $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Substituting the co-ordinates of the pt. $P'(-x, -y)$ in the equation of the hyperbola, we get $\frac{(-x)^2}{a^2}-\frac{(-y)^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which is true, $\therefore$ P′ also lies on the hyperbola.

Now the mid-pt. of PP′ is

$$\left[\frac{x+(-x)}{2}, \frac{y+(-y)}{2}\right] \text{ or } (0, 0), \textit{ i.e.}, \text{ C.}$$

Cor. The **equation of a rectangular hyperbola** is [putting $b=a$]

$$\frac{x^2}{a^2}-\frac{y^2}{a^2}=1 \text{ or } \mathbf{x^2-y^2=a^2}.$$

EXAMPLES

1. (*a*) **Obtain the equation of a hyperbola referred to its principal axes.** [*B. U. 1938*]

(*b*) Find the equation to the hyperbola, referred to its axes as axes of co-ordinates, whose transverse and conjugate axes are respectively 2 and 3.

2. Find the equation to the hyperbola, referred to its axes as axes of co-ordinates, whose conjugate axis is 5 and which passes through the point (1, −2).

3. Equation of a hyperbola referred to any focus and directrix. Find the equation to the hyperbola whose eccentricity is 2, whose focus is (2, 0) and whose directrix is x−y=0. [*P.U.*]

140. Tracing a hyperbola. *To trace the curve* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The equation of the curve is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)

(*i*) **Axes-intersections.** The x-axis meets the curve where, putting $y=0$ in (1), $\frac{x^2}{a^2}=1$ or $x^2=a^2$ or $x=\pm a$, *i.e.*, in the pts. A$(a, 0)$, A$'(-a, 0)$. (See first Fig. on p. 229.)

Again, the y-axis meets the curve where, putting $x=0$ in (1),

$$-\frac{y^2}{b^2}=1 \text{ or } y^2=-b^2 \text{ or } y=\pm ib \text{ (imaginary)},$$

$\therefore$ the y-axis does not meet the curve (in real pts.).

(*ii*) **Symmetry.** From (1), $\frac{y^2}{b^2}=\frac{x^2}{a^2}-1=\frac{x^2-a^2}{a^2}$

or $$y=\pm\frac{b}{a}\sqrt{x^2-a^2} \quad ...(2)$$

$\therefore$ for every value of x numerically $> a$, there are two equal and opposite values of y, $\therefore$ the curve is symmetrical about the x-axis.

Again from (1), $\frac{x^2}{a^2}=1+\frac{y^2}{b^2}=\frac{b^2+y^2}{b^2}$

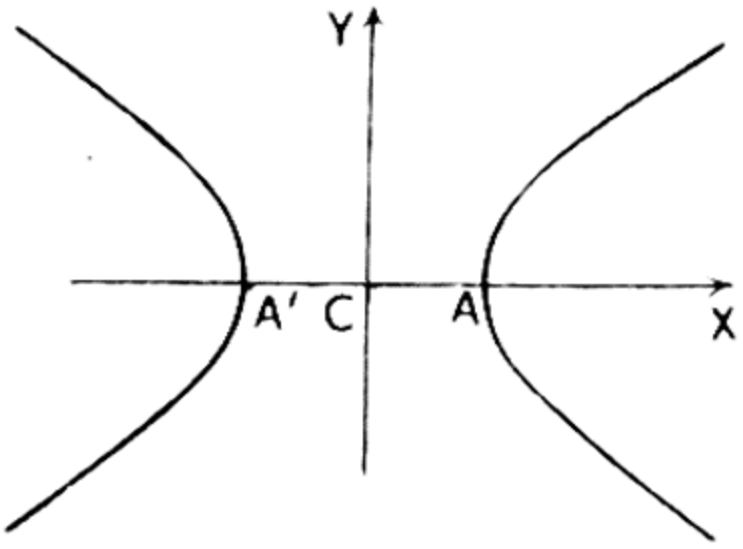
or $$x=\pm\frac{a}{b}\sqrt{b^2+y^2} \quad ...(3)$$

$\therefore$ for every value of y there are two equal and opposite values of x, $\therefore$ the curve is symmetrical about the y-axis.

(*iii*) **Imaginary values.** From (2), y is imaginary when x is numerically $< a$, $\therefore$ no portion of the curve lies between A and A′.

(*iv*) **Space occupied.** From (2), taking +ve sign with the square root, $y=\frac{b}{a}\sqrt{x^2-a^2}$, $\therefore$ as x (numerically $> a$) increases, y increases; when $x=\infty$, $y=\infty$, $\therefore$ the curve extends to ∞, to the right of A and to the left of A′, *i.e.*, it is open.

Hence the shape of the curve is that shown in the Fig. It consists of two infinite branches.

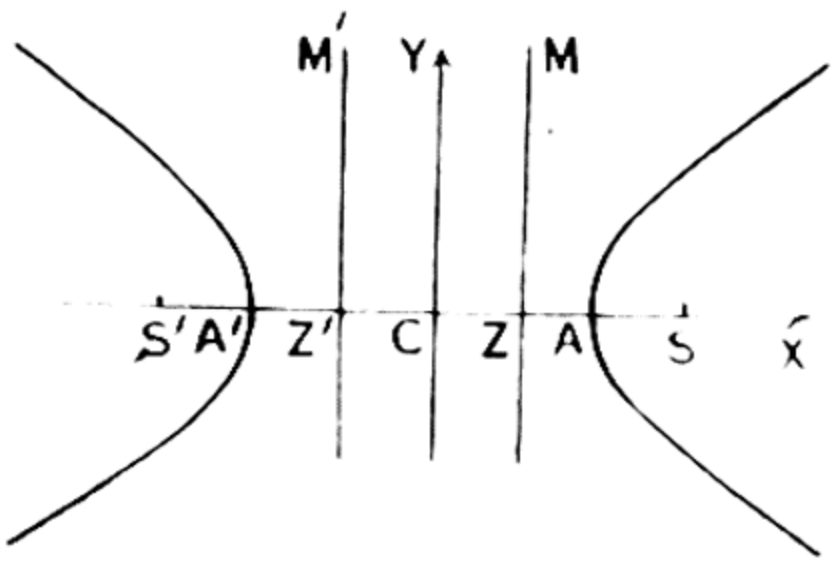


[**Rule to trace a hyperbola whose equation is given :** *See Rule of Art. 96.*]

[**Aid to memory.** See Art. 96, **Aid to memory.**]

Cor. The hyperbola has a second focus and a second directrix.

Proof. ∵ the hyperbola is symmetrical about the y-axis (Art. 140, (*ii*)), ∴ if on the x-axis the pts. S′, Z′ are taken such that CS′=CS, CZ′=CZ, then the pt. S′ will also be a focus of the hyperbola and Z′M′, the line thro' Z′ ⊥ to CZ′, will be the *corresponding* directrix. [See Fig. below.]



[**Important.** *The foci S, S′ lie on the transverse axis on opposite sides of C such that* **CS=CS′=ae** ; *the corresponding directrices ZM, Z′M′ are perpendicular to the transverse axis on opposite sides of C such that* $\mathbf{CZ=CZ'=\frac{a}{e}}$.

N. B. *The foci of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *are* ($\pm$**ae, 0**), *and the equations of the corresponding directrices are* $\mathbf{x=\pm\frac{a}{e}}$.]

EXAMPLES

Find the equation of the hyperbola, referred to its axes as axes of co-ordinates (Exs. 1—2) :

1. (*i*) whose foci are (2, 0) and (−2, 0) and eccentricity $\frac{3}{2}$.

[*P. U. 1954*]

(*ii*) the distance between whose foci is 4, and whose eccentricity is $\sqrt{2}$.

2. whose conjugate axis is 3 and the distance between whose foci is 5.

3. Find the eccentricity of the hyperbola whose equation is $2x^2-3y^2=15$. [*P.U. 1955*]

4. Find the eccentricity and the co-ordinates of the foci of the curve $3x^2-y^2=4$. [*A. U. 1941*]

****5. Trace the curve $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.**

[**Hint.** The equation of the curve is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ ...(1)

Proceed as in Art. 140. Here in step (*i*) *Axes-intersections*, the x-axis meets the curve where, putting $y=0$ in (1),

$-\frac{x^2}{b^2}=1$ or $x^2=-b^2$ or $x=\pm ib$ (imaginary),

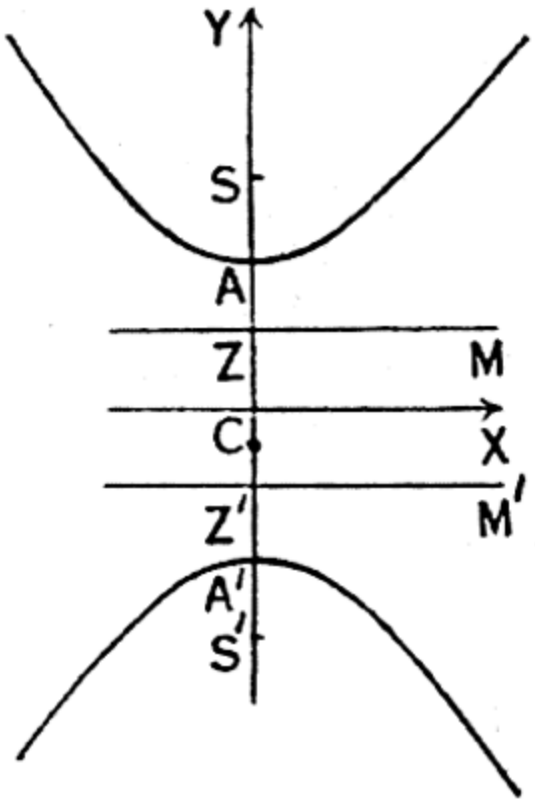
$\therefore$ the x-axis does not meet the curve (in real pts.).

Again, the y-axis meets the curve where, putting $x=0$ in (1),

$\frac{y^2}{a^2}=1$ or $y^2=a^2$ or $y=\pm a$,

i.e., in the pts. A$(0, a)$, A$'(0, -a)$.

$\therefore$ the transverse axis $(=2a)$ of the hyperbola lies along the *y-axis*.



Cor. To find the foci and the directrices of the hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

The foci lie on the *transverse* axis on opposite sides of C such that $CS=CS'=ae$. $\therefore$ the foci are $(0, \pm ae)$.

Again, the directrices ZM, Z'M' are $\perp$ to the transverse axis on opposite sides of C such that $CZ=CZ'=\frac{a}{e}$.

$\therefore$ the equations of the corresponding directrices are $y=\pm\frac{a}{e}$.

Note. First and second standard forms. The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1) of a hyperbola in which the coefficient of y^2 is negative, is called the **first standard form,** and (changing x to y and y to x in (1)), $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ ...(2) in which the coefficient of x^2 is negative, is called the **second standard form.**

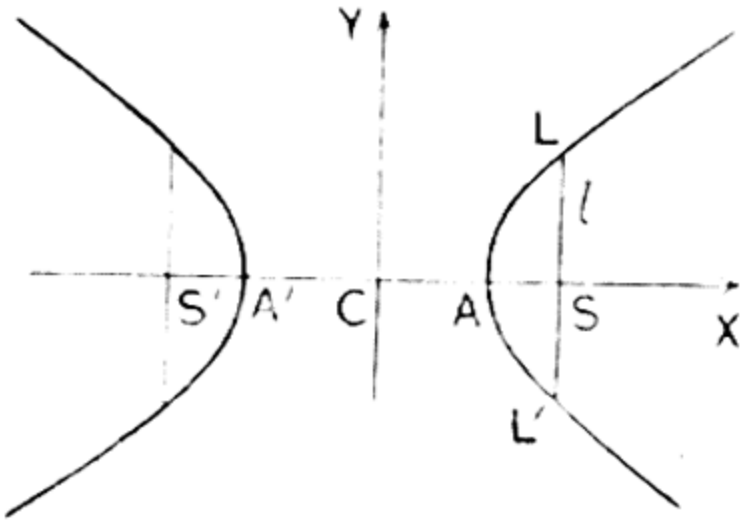
Important. *To write down the standard results, the student should fix in his memory the positions (Figs.) of the hyperbolas with respect to the co-ordinate axes.*]

[Aid to memory. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in which the coefficient of y^2 is negative, is a hyperbola with its transverse axis horizontal.

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ in which the coefficient of x^2 is negative, is a hyperbola with its transverse axis vertical.]

****6.** Find the co-ordinates of the foci of the curve $9y^2-4x^2=36$.

141. Length of the latus rectum. *To find the length of the latus rectum of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.



The equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Let LSL′ be the latus rectum, and $SL=l$. Then the pt. L(ae, l) lies on the hyperbola

$$\therefore \quad \frac{a^2e^2}{a^2}-\frac{l^2}{b^2}=1 \quad \text{or} \quad \frac{l^2}{b^2}=e^2-1$$

or $\quad l^2=b^2(e^2-1) \qquad [\text{ But } b^2=a^2(e^2-1), \quad \therefore \quad e^2-1=\frac{b^2}{a^2} \text{ }]$

$$=b^2.\frac{b^2}{a^2}=\frac{b^4}{a^2}$$

$$\therefore \quad l=\frac{b^{2*}}{a},$$

$$\therefore \quad \text{LSL}'=2l=2\,\frac{\mathbf{b^2}}{\mathbf{a}}.$$

[**N.B.** *The latus rectum of each of the hyperbolas* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ *is* $2\,\frac{\mathbf{b^2}}{\mathbf{a}}$.

The result may be proved in the second case by drawing the Fig. (Art. 140, Ex. 5) and proceeding as in Art. 141. Here L(l, ae) lies on the hyperbola.]

EXAMPLES

1. In the hyperbola $x^2-4y^2=4$, find the axes, the co-ordinates of the foci, the eccentricity, and the latus rectum. [*P. U.*]

Find also the equations of the directrices. [*P.U. 1954S*]

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* ∵ $l=SL$ which, from the Fig., is +ve.

2. Find the eccentricity, the distance between the foci, and the latus rectum in the curve $x^2-y^2=4$.

SECTION II
TANGENTS AND NORMALS

142. Most of the results for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ may be proved by the methods employed for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

For, in the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, changing b^2 to $-b^2$, we get $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which is the equation of the hyperbola.

Hence we shall leave the proofs of these results as exercises to the student.

[Aid to memory for standard results for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

In the corresponding results for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, change b^2 to $-b^2$.

(For a point of contrast, see end of Art. 143.)]

EXAMPLE

Find the equation of the tangent at the point (x_1, y_1) of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. [*B.U. 1936*]

Find also the equation of the normal. [*B.H.U. 1945*]

[**Hint.** Proceed as in Arts. 122, 123.]

[**N.B.** The equation of the tangent at the point (x_1, y_1) of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

The equation of the normal is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.]

[**Rule to write down the equation of the tangent at the point (x_1, y_1) of the hyperbola (equation in the standard form) :**
See Rule of Art. 64.]

[**Rule to find the equation of the tangent at the point (x_1, y_1) of the hyperbola (equation in the standard form) :** *See Rule of Art. 63.*]

SECTION III
A HYPERBOLA AND A LINE
EXAMPLES

1. Points of intersection. Show how to find the points of intersection of the line $y=mx+c$ with the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

[**Hint.** Proceed as in Art. 124.
It will be found that the quadratic in x is
$x^2(a^2m^2-b^2)+2a^2mcx+a^2(c^2+b^2)=0$.]

2. Length of the intercept. Find the length of the intercept made by the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ on the line $y=mx+c$, and deduce the condition that the line may touch the hyperbola.

[**Hint.** Proceed as in Art. 125 and Cor.]

3. (*a*) Find the points common to the hyperbola $4x^2-4y^2=15$ and the straight line $x-2y-3=0$.

(*b*) Find the length of the straight line intercepted by the hyperbola.

4. Prove that the straight line $5x+12y=9$ touches the hyperbola $x^2-9y^2=9$, and find its point of contact. [*P. U.*]

5. Condition of tangency. Find the condition that the line $y=mx+c$ may touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. [*M.U. 1943S*]

Cor. 1. Equation of a tangent in terms of its slope. *If m is the slope of a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the equation of the tangent is* $y=mx\pm\sqrt{a^2m^2-b^2}$.

Cor. 2. Any tangent in the m-form. *The equation of any tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, in the m-form, is* $y=mx+\sqrt{a^2m^2-b^2}$.

[**Hint.** Proceed as in Art. 126 and Cors. 1 and 2.]

6. Find the equations of the tangents to the hyperbola $3x^2-4y^2=15$ which are parallel to $y=2x+k$. [*M.U.*]

7. Find the equations of the tangents to the hyperbola $3x^2-y^2=3$ which are

(*i*) parallel to the line $y=2x+4$.

(*ii*) perpendicular to the line $x+3y=2$. [*P. U. 1954S*]

8. Show that the line $y=mx+\sqrt{a^2m^2-b^2}$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for all values of m. [*P.U. 1956*]

9. Condition of tangency. Show that the line $lx+my+n=0$ will touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $a^2l^2-b^2m^2=n^2$.

[**Short-cut.** Proceed as in Ex. 9, Art. 71.]

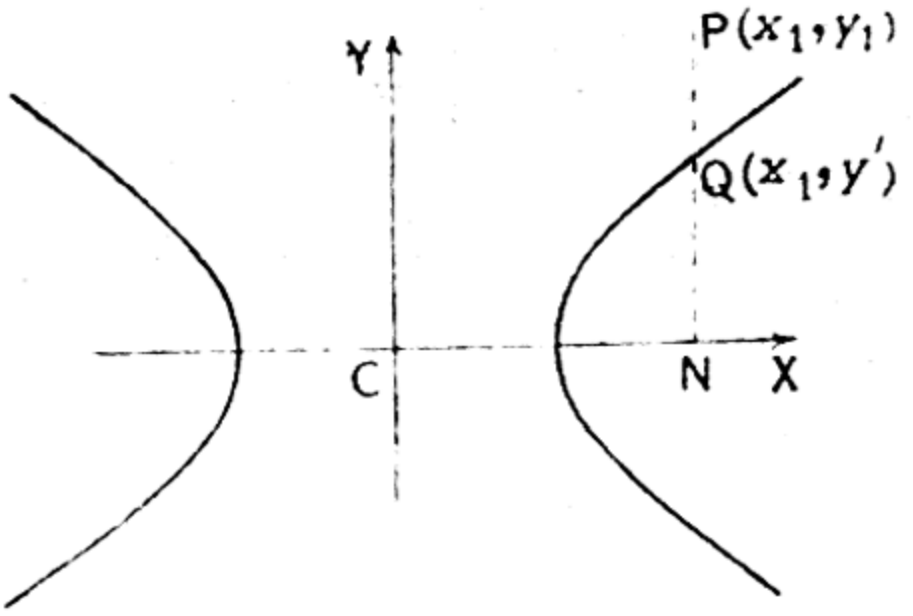
10. Show that the line $x\cos\alpha+y\sin\alpha=p$ will touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if $p^2=a^2\cos^2\alpha-b^2\sin^2\alpha$. [*B.H.U. 1949*]

SECTION IV

A HYPERBOLA AND A POINT

143. Test for a point to lie outside, on, or inside a hyperbola. *To show that the quantity $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$* **is negative, zero, or positive**

according as the point (x_1, y_1) *lies outside, on, or inside the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. [Contrast with Arts. 72, 107, 127]



The equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Let P be the pt. (x_1, y_1). From P draw PN $\perp$ on the x-axis to meet the hyperbola in $Q(x_1, y')$.

$\because$ Q lies on the hyperbola

$\therefore \quad \frac{x_1^2}{a^2}-\frac{y'^2}{b^2}=1$

or $\frac{x_1^2}{a^2}-\frac{y'^2}{b^2}-1=0$ ...(1) [R.H.S. zero]

Now $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1-0$ [Substitute for 0 from (1)]

$$=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1-\left(\frac{x_1^2}{a^2}-\frac{y'^2}{b^2}-1\right)$$

$$=\frac{1}{b^2} \quad [\ y'^2-y_1^2\ ]$$

$$=\frac{1}{b^2} \quad [\ NQ^2-NP^2\ ] \quad ...(2)$$

(*i*) If P lies outside the hyperbola (as in the Fig.), NP>NQ,

$\therefore$ from (2)*, $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is $-$ve.

(*ii*) If P lies on the hyperbola, NP=NQ,

$\therefore$ from (2), $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is zero.

(*iii*) If P lies inside the hyperbola, NP<NQ,

$\therefore$ from (2), $\frac{x_1^2}{a^2}-\frac{y_1^3}{b^2}-1$ is $+$ve.

A point of contrast. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the quantity

---

* $\because$ R.H.S. is $-$ve, $\therefore$ L.H.S. is also $-$ve.

$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ is positive, zero, or negative according as the point (x_1, y_1) lies outside, on, or inside the ellipse (Art. 127).

But for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the quantity $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is *negative, zero, or positive* according as the point (x_1, y_1) lies outside, on, or inside the hyperbola.

[****Rule to find whether a point lies outside, on, or inside a hyperbola (equation in the standard form) :** *In the L.H.S. of the equation of the curve (R.H.S. being zero), substitute the co-ordinates of the point ; if the result is negative, the point lies outside the curve, if zero, it lies on the curve, and if positive, it lies inside the curve. Contrast this Rule with that to find whether a point lies outside, on, or inside a circle, parabola or ellipse (equation in the standard form) (Arts. 72, 107, 127).*]

EXAMPLES

1. Explain the significance of $P=\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1$ in relation to a point (x', y') if (*i*) $P=0$, (*ii*) $P>0$, and (*iii*) $P<0$. [*B.H.U. 1946*]

2. Two tangents from a point. Show that from any point two tangents can be drawn to a hyperbola, and they are real, coincident, or imaginary according as the point is outside, on, or inside the hyperbola.

[**Hint.** Proceed as in Art. 128.]

3. Tangents are drawn from the point $(-2, -1)$ to the hyperbola $2x^2-3y^2=6$. Find their equations. [*P. U. 1954*]

4. Tangents are drawn to the hyperbola $5x^2-y^2=5$ from the point $(0, 2)$. Find their equations.

5. Director circle. Find the locus of the point of intersection of two perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. [*P. U. 1956*]

[**Hint.** Proceed as in Art. 129.

****Note.** It will be found that the locus is $x^2+y^2=a^2-b^2$, which is a circle. It is called the **director circle** of the hyperbola. Its centre is the origin, and radius $=\sqrt{a^2-b^2}$. Hence :

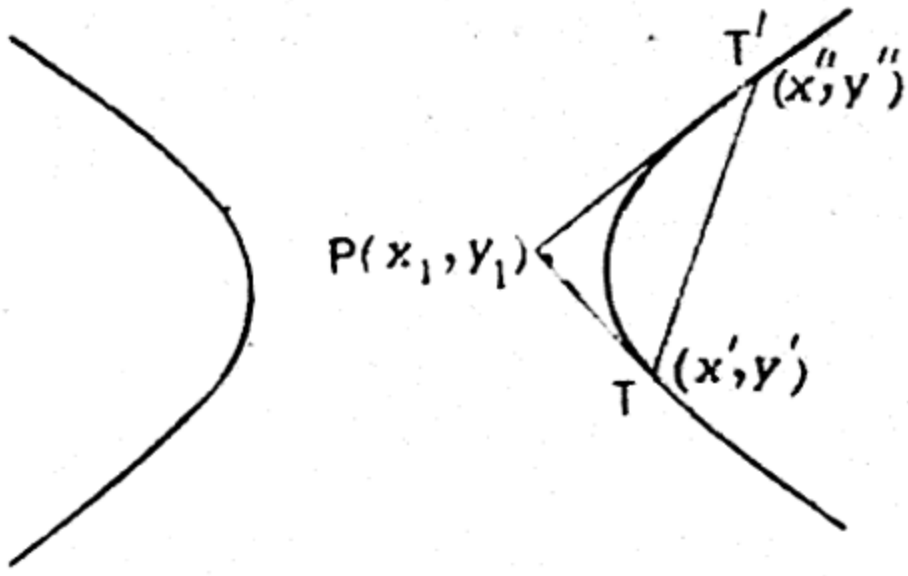
(*i*) If $a^2>b^2$, the radius is real, $\therefore$ the director circle is real.

(*ii*) If $a^2=b^2$, the radius $=0$, $\therefore$ the director circle reduces to a point circle at the origin. In this case the origin, *i.e.*, the centre of the hyperbola is the only point from which two perpendicular tangents can be drawn to the hyperbola.

(*iii*) If $a^2<b^2$, the radius is imaginary, $\therefore$ the director circle is imaginary. In this case there is no point from which two perpendicular tangents can be drawn to the hyperbola.]

Chord of contact

6. Equation of the chord of contact. Find the equation of the chord of contact of tangents from (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.



[**Hint.** Proceed as in Art. 130.]

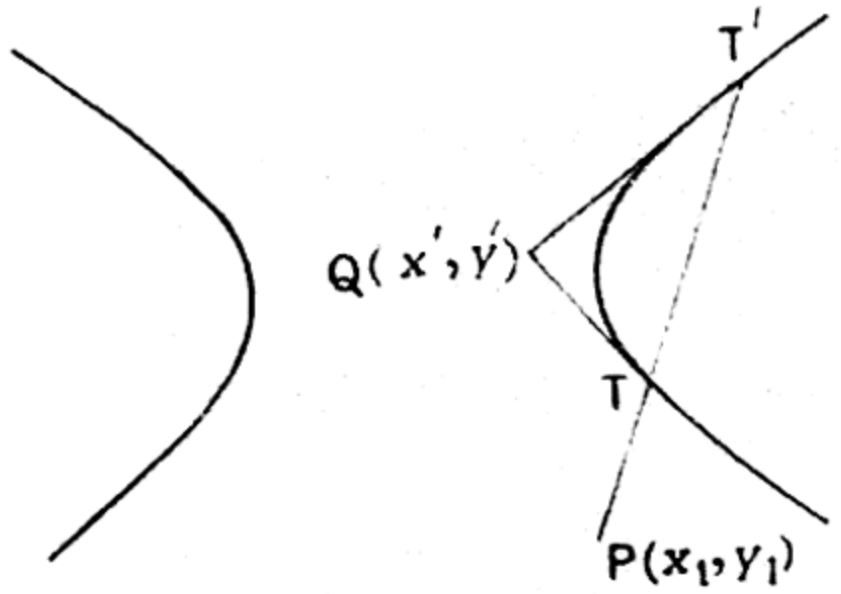
[**N.B.** *The equation of the chord of contact of tangents from* (x_1, y_1) *to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *is* $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.]

Pole and polar

7. (*a*) Define the polar of a point with respect to a hyperbola. [*M.U. 1937*]

(*b*) **Equation of the polar. Find the equation of the polar of the point** (x_1, y_1) **with respect to the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^3} = 1$.

(*c*) **Show that the polar of the focus w. r. t. a hyperbola is the corresponding directrix.**



[**Hint.** (*b*) Proceed as in Art. 131.]

[**N. B.** *The equation of the polar of* (x_1, y_1) *w. r. t. the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *is* $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.]

[**Aid to memory.** See Art. 79, **Aid to memory.**]

8. Pole of a given line. Find the pole of the line $Ax+By+C=0$ **with respect to the hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. [*M. U. 1937*]

[**Hint.** Proceed as in Art. 132.]

9. Find the pole of the chord $x-2y=1$ **with respect to the hyperbola** $4x^2-9y^2=12$.

10. The line $Ax+By+C=0$ **meets the hyperbola** $4x^2-y^2=4a^2$ **in the points P and Q. Find the co-ordinates of the point of intersection of the tangents at P and Q.** [*P. U. 1956 S*]

Properties of pole and polar

11. Reciprocal property. If the polar of a point P with respect to a hyperbola passes through a point Q, then the polar of Q passes through P. [*P. U. 1955 S*]

[**Hint.** Proceed as in Art. 133.]

12. If the polars of two points P and Q with respect to a hyperbola meet in R, then R is the pole of the line PQ.

Use of parameter

[**Auxiliary circle. Def.** The circle described on the transverse axis, AA′, of a hyperbola as diameter, is called the **auxiliary circle** of the hyperbola.]

144. *To express the co-ordinates of any point on the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *in terms of one variable* (*called the* **parameter**).

The equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \ldots(1)$$

Also $$\sec^2\theta-\tan^2\theta=1 \qquad \ldots(2)$$

∴ comparing (1) and (2),

$$\frac{x^2}{a^2}=\sec^2\theta,\ \frac{y^2}{b^2}=\tan^2\theta$$

or

$$\frac{x}{a}=\sec\theta,\ \frac{y}{b}=\tan\theta$$

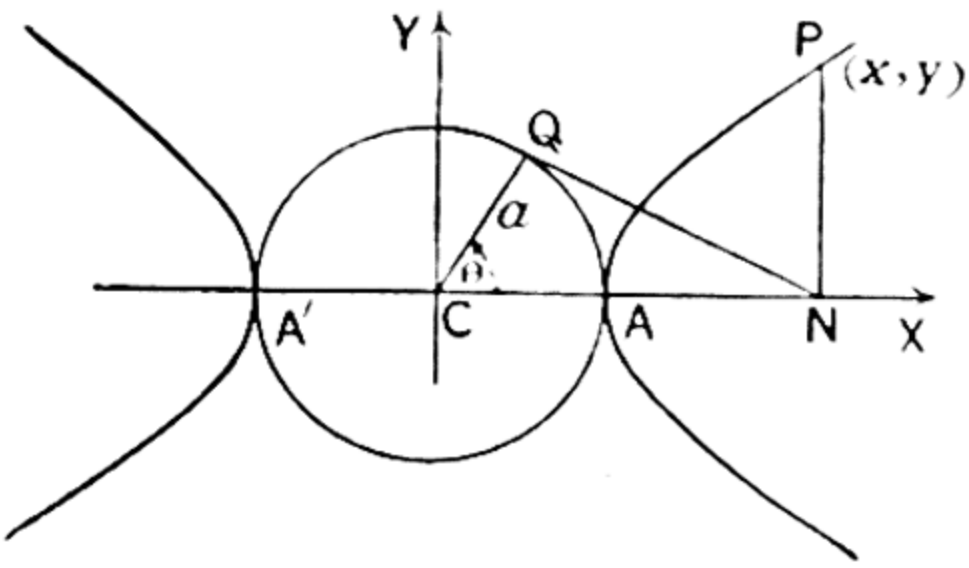
∴ $\mathbf{x=a\sec\theta}$, $\mathbf{y=b\tan\theta}$, which are the required co-ordinates.

****Another method (Geometrically).**

The equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Let P (x, y) be *any* pt. on the hyperbola, and NP its ordinate.

From N draw NQ tangent to the auxiliary circle to touch it at Q, and join CQ.



Let $\angle ACQ=\theta$.

Then $x=CN$

$$\left[\text{ But } \frac{CN}{CQ}=\sec\theta, \quad \therefore \quad CN=CQ\sec\theta \right]$$

$=a\sec\theta$. [∵ $CQ=CA=a$]

Substituting this value of x in the equation of the hyperbola,

$\frac{a^2\sec^2\theta}{a^2}-\frac{y^2}{b^2}=1$, or $\frac{y^2}{b^2}=\sec^2\theta-1=\tan^2\theta$, or $y^2=b^2\tan^2\theta$

∴ $y=b\tan\theta$.*

∴ $\mathbf{x=a\sec\theta}$, $\mathbf{y=b\tan\theta}$, which are the required co-ordinates.

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* ∵ $y=NP$ which, from the Fig., is +ve.

[**Note.** $\angle$ACQ in the hyperbola corresponds to the eccentric angle in the ellipse ; but it is not so important as that angle, and has no special name.

Comparison of the two methods. The first method does not give the geometrical meaning of θ, *i.e.*, $\angle ACQ=\theta$.]

Cor. Any point on the hyperbola. *Any point on the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *is* **(a sec θ, b tan θ).**

Note. Since the position of the point P (a sec θ, b tan θ) is known when the value of θ is given, hence the

Def. *The point whose co-ordinates are* (a *sec* θ, b *tan* θ) *is, for shortness, called the* "**point θ**".

EXAMPLES

1. Show that (a sec θ, b tan θ) represents a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. [*P. U. 1955S*]

P is a variable point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose vertex A is (a, 0). Show that the locus of the middle point of AP is

$$\frac{(2x-a)^2}{a^2}-\frac{4y^2}{b^2}=1.$$ [*P. U. 1953S*]

2. Find the equation of the straight line joining two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with parameters θ_1 and θ_2.

[*B.U. 1947*]

Deduce the equation of the tangent at the point θ_1.

[**Hint.** (*i*) Proceed as in Art. 136, and express secants and tangents in terms of the sine and cosine.

It will be found that the equation of the line passing thro' the two pts. is

$$\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}$$

$$=\frac{1}{\cos\theta_1}\cos\frac{\theta_1-\theta_2}{2}-\frac{\sin\theta_1}{\cos\theta_1}\sin\frac{\theta_1+\theta_2}{2}$$

$$=\frac{1}{\cos\theta_1}\left[\cos\frac{\theta_1-\theta_2}{2}-\sin\theta_1\sin\frac{\theta_1+\theta_2}{2}\right]$$

$$=\frac{1}{\cos\theta_1}\left[\cos\left(\theta_1-\frac{\theta_1+\theta_2}{2}\right)-\sin\theta_1\sin\frac{\theta_1+\theta_2}{2}\right]$$ [*Note this step*]

$$=\frac{1}{\cos\theta_1}\left[\cos\theta_1\cos\frac{\theta_1+\theta_2}{2}+\sin\theta_1\sin\frac{\theta_1+\theta_2}{2}-\sin\theta_1\sin\frac{\theta_1+\theta_2}{2}\right]$$

$$=\cos\frac{\theta_1+\theta_2}{2}.$$

(*ii*) It will be found that the equation of the tangent at (a sec θ_1, b tan θ_1) to the hyperbola is $\frac{x}{a}\sec\theta_1-\frac{y}{b}\tan\theta_1=1$.]

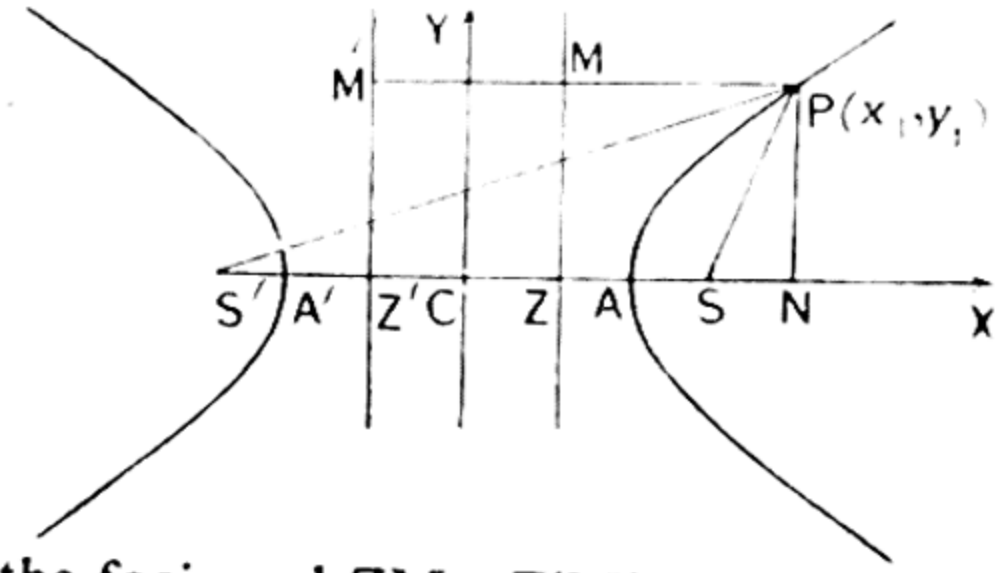
3. Find the equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (a sec θ, b tan θ). [*P.U. 1956S*]

[It will be found that the equation of the normal at the point (a sec θ, b tan θ) is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$.]

SECTION V

GEOMETRICAL PROPERTIES

145. Focal property. The difference of the focal distances of any point on a hyperbola is constant, and equal to the transverse axis.*



Let S, S′ be the foci, and ZM, Z′M′ the corresponding directrices of the hyperbola.

Let P(x_1, y_1) be *any* pt. on the hyperbola. Join SP, S′P.

Thro' P draw a st. line PMM′ ∥ to the x-axis to meet ZM, Z′M′ in M, M′. From P draw PN ⊥ on the x-axis.

(*i*) $SP=e.\ PM=e.\ NZ=e(CN-CZ)=e\left(x_1-\frac{a}{e}\right)=ex_1-a$ ...(1)

(*ii*) $S'P=e.PM'=e.NZ'=e(CN+CZ')=e\left(x_1+\frac{a}{e}\right)=ex_1+a$ ...(2)

$\therefore\ S'P-SP=ex_1+a-(ex_1-a)=ex_1+a-ex_1+a=2a$ (constant)

=transverse axis.

Cor. Focal distances. *The focal distances of any point* (x_1, y_1) *on the hyperbola are* $\mathbf{ex_1\pm a}$.† [Cf. Art. 137,(*a*). Cor.]

Note. The property of a hyperbola proved in Art. 145 is extremely *important*, and is sometimes taken as its

Def. *A* **hyperbola** *is the locus of a point which moves so that the difference of its distances from two fixed points is constant.*

The fixed points are called the **foci** of the hyperbola, and are denoted by S, S′.

EXAMPLES

1. Derive the equation of a hyperbola by defining it as the locus of a point P such that SP∼‡S′P=2a, where S and S′ are two fixed points whose distance is greater than 2a.

[‡**Notation.** SP∼S′P denotes the difference between SP and S′P.]

*[*P. U. 1955*] †[*P. U. 1942*]

2. (*a*) If the tangent at any point P of a hyperbola meets the transverse axis in T, and PN is perpendicular on the transverse axis, then

$$CT.CN = CA^2.$$

(*b*) If the tangent at any point P of a hyperbola meets the conjugate axis in *t*, and P*n* is perpendicular on the conjugate axis, then

$$Ct.Cn = CB^2.*$$

3. If the normal at any point P of a hyperbola meets the transverse axis in G, and PN is perpendicular on the transverse axis, then

$$\mathbf{CG = e^2.CN.}$$

[**N.B.** If the normal at any point $P(x_1, y_1)$ of the hyperbola meets the transverse axis in G, then $\mathbf{CG = e^2.x_1}$.]

4. The tangent at any point of a hyperbola bisects the angle between the focal distances of the point, and the normal bisects the exterior angle between the focal distances.

****5.** If the normal at any point P of a hyperbola with centre C and foci S, S′ meets the transverse and conjugate axes in G and *g*, and if CF be the perpendicular upon the normal, prove that $PF.PG = b^2$, $PF.Pg = a^2$, where *a* is the semi-axis transverse and *b* the semi-axis conjugate.

6. The locus of the feet of the perpendiculars from the foci on any tangent to a hyperbola is the auxiliary circle, and the product of the perpendiculars is equal to the square on the semi-conjugate axis.

7. Diameter property. The locus of the middle points of a system of parallel chords of a hyperbola is a straight line passing through the centre.

[**N.B.** *The equation of the diameter of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *bisecting parallel chords of slope m, is* $\mathbf{y = \frac{b^2}{a^2 m}\, x.}$]

8. Find the equation of the locus of the middle points of the chords of the hyperbola $2x^2 - 3y^2 = 1$, each of which makes an angle of 45° with the *x*-axis. [*M. U. 1936S*]

****9. Equation of a chord in terms of its mid-point. The equation of the chord of the hyperbola** $\frac{\mathbf{x^2}}{\mathbf{a^2}} - \frac{\mathbf{y^2}}{\mathbf{b^2}} = \mathbf{1}$, **whose mid-point is** $(\mathbf{x_1, y_1})$, **is** $\frac{\mathbf{xx_1}}{\mathbf{a^2}} - \frac{\mathbf{yy_1}}{\mathbf{b^2}} = \frac{\mathbf{x_1^2}}{\mathbf{a^2}} - \frac{\mathbf{y_1^2}}{\mathbf{b^2}}$.

[**Hint.** Proceed as in Method I, solved Ex. 1, Art. 84. (Here the mid-pt. (x_1, y_1) lies on the diameter $y = \frac{b^2}{a^2 m}\, x$ (Ex. 7).]

[**N.B.** *The equation of the chord of the hyperbola* $S = 0$, *whose mid-point is* (x_1, y_1), *is* $\mathbf{T = S_1}$, *where* $T = 0$ *is the equation of the tangent at* (x_1, y_1), *and* S_1 *is the result of substituting the co-ordinates of the point* (x_1, y_1) *in* S.]

****10.** Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$, which is bisected at the point (5, 3). [*Pesh. U. 1953*]

*It will be found that $Ct.Cn = -b^2$. The negative sign shows that C*t* and C*n* are drawn in opposite directions. Similarly for Ex. 6, (*ii*).

Deduce the condition that $lx+my+n=0$ may touch the ellipse. [*P. U.*]

[**Sol.** (*i*) The equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

Substituting the co-ordinates of the pt. $(a\cos\theta, b\sin\theta)$ in (1), we get $$\frac{a^2\cos^2\theta}{a^2}+\frac{b^2\sin^2\theta}{b^2}=1$$

or $\cos^2\theta+\sin^2\theta=1$, which is true, $\therefore$ the pt. lies on the ellipse.

(*ii*) θ is the eccentric angle of the pt.

(*iii*) The equation of the tangent at $(a\cos\theta, b\sin\theta)$ is $$\frac{x(a\cos\theta)}{a^2}+\frac{y(b\sin\theta)}{b^2}=1 \qquad \left[\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1\right]$$

or $$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

(*iv*) The equation of the tangent at $(a\cos\theta, b\sin\theta)$ is $$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \text{ or } \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}-1=0 \quad ...(1)$$

If it is the same as the equation of the given line $$lx+my+n=0 \quad ...(2)$$

then comparing coefficients in (1) and (2),

$$\frac{\frac{\cos\theta}{a}}{l}=\frac{\frac{\sin\theta}{b}}{m}=-\frac{1}{n} \text{ or } \frac{\cos\theta}{al}=\frac{\sin\theta}{bm}=-\frac{1}{n},$$

$\therefore \quad \cos\theta=-\frac{al}{n}, \quad \sin\theta=-\frac{bm}{n}.$

Squaring and adding, $1=\frac{a^2l^2}{n^2}+\frac{b^2m^2}{n^2}$ $(\because \cos^2\theta+\sin^2\theta=1)$

or $\mathbf{a^2l^2+b^2m^2=n^2}$, which is the required condition.]

2. Find the point of intersection of the tangents at the points θ_1, θ_2 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

[**Hint.** Write down the equations of the tangents, and solve them for $\frac{x}{a}$, $\frac{y}{b}$ by cross-multiplication as in Misc. Ex. 15, (*ii*), Chap. III.

****Short-cut. Hint.** The equation of the line, joining the pts. θ_1, θ_2 on the ellipse, is

$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}. \qquad \text{(Art. 136)}$$

Find its pole w.r.t. the ellipse.]

3. Define the eccentric angle of a point on an ellipse. [*P(P). U. 1950*]

Find the equation of the normal at any point of an ellipse whose eccentric angle is given. [*Pesh. U. 1955*]

[**Sol.** (*ii*) Let θ be the eccentric angle of the pt. Then its co-ordinates are $(a\cos\theta, b\sin\theta)$.

The equation of the normal at $(a\cos\theta, b\sin\theta)$ is

$$\frac{a^2x}{a\cos\theta}-\frac{b^2y}{b\sin\theta}=a^2-b^2 \qquad \left[\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2\right]$$

or $$\frac{\mathbf{ax}}{\cos\theta}-\frac{\mathbf{by}}{\sin\theta}=\mathbf{a^2-b^2}.\ ]$$

4. Prove that the line $lx+my+n=0$ is a normal to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ if } \frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}. \qquad [Pesh.\ U.\ 1955]$$

[**Sol.** The equation of the normal at $(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \text{ or } \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}-(a^2-b^2)=0 \quad \ldots (1)$$

If it is the same as the equation of the given line $lx+my+n=0$ ... (2)

then comparing coefficients in (1) and (2),

$$\frac{a}{\cos\theta}=\frac{b}{\sin\theta}=\frac{-(a^2-b^2)}{n} \text{ or } \frac{a}{l\cos\theta}=-\frac{b}{m\sin\theta}=-\frac{a^2-b^2}{n}$$

$$\therefore \cos\theta=-\frac{an}{l(a^2-b^2)},\ \sin\theta=\frac{bn}{m(a^2-b^2)}.$$

Squaring and adding, $1=\dfrac{a^2n^2}{l^2(a^2-b^2)^2}+\dfrac{b^2n^2}{m^2(a^2-b^2)^2}$

$(\because \cos^2\theta+\sin^2\theta=1)$

or $\dfrac{a^2}{l^2}+\dfrac{b^2}{m^2}=\dfrac{(a^2-b^2)^2}{n^2}$, which is the required condition.]

5. Prove that the line $y=mx+c$ is a normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

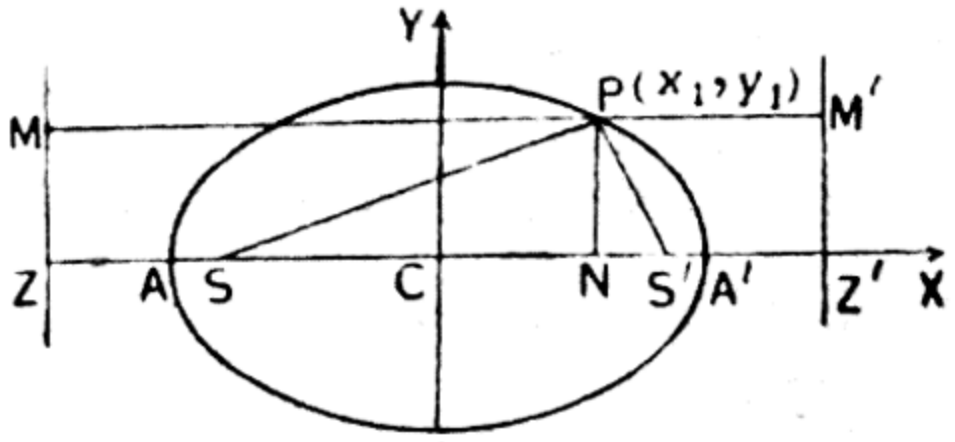
if $$\frac{a^2}{m^2}+b^2=\frac{(a^2-b^2)^2}{c^2}. \qquad [\ P.\ U.\ 1949\ ]$$

6. (*i*) **Find the equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, joining the points whose eccentric angles are $\alpha+\beta$ and $\alpha-\beta$.** [*C. U. 1947*]

(*ii*) **The eccentric angles of two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\frac{\pi}{3}$ and $\frac{\pi}{6}$. Find the equation of the chord passing through them.** [*P(P). U. 1952*]

GEOMETRICAL PROPERTIES

137. (*a*) Focal property. The sum of the focal distances of any point on an ellipse is constant, and equal to the major axis.*



*[*B.H.U. 1951*]

Let S, S′ be the foci, and ZM, Z′M′ the corresponding directrices of the ellipse.

Let P(x_1, y_1) be *any* pt. on the ellipse. Join SP, S′P.

Thro' P draw a st. line MPM′ ∥ to the x-axis to meet ZM, Z′M′ in M, M′. From P draw PN ⊥ on the x-axis.

(*i*) $SP = e.PM = e.NZ = e(CZ+CN) = e\left(\frac{a}{e} + x_1\right) = a + ex_1$ ...(1)

(*ii*) $S'P = e.PM' = e.NZ' = e(CZ'-CN) = e\left(\frac{a}{e} - x_1\right) = a - ex_1$ ...(2)

$\therefore$ $SP + S'P = a + ex_1 + a - ex_1 = 2a$ (constant)
= major axis.

Cor. Focal distances. *The focal distances of any point* (x_1, y_1) *on the ellipse are* $\mathbf{a \pm ex_1}$.

Note. The property of an ellipse proved in Art. 137, (*a*) is extremely *important*, and is sometimes taken as its

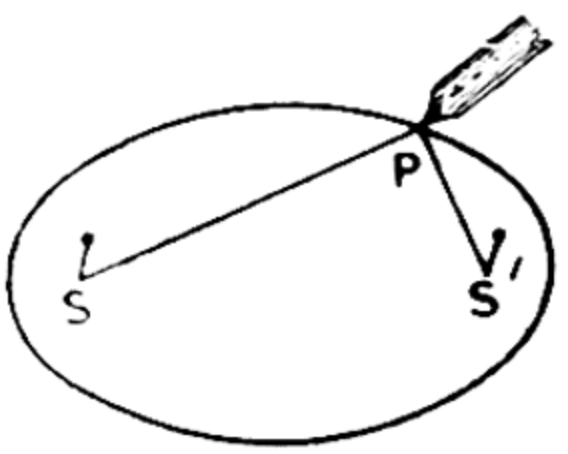
Def. *An* **ellipse** *is the locus of a point which moves so that the sum of its distances from two fixed points is constant.*

The fixed points are called the **foci** of the ellipse, and are denoted by S, S′.

EXAMPLES

1. Mechanical construction for ellipse. Give a mechanical construction for an ellipse. [*A. U. 1938*]

[**Construction.** Fix two pins at the foci S, S′. Take a thread of length $2a$(=major axis), and fasten its ends to S, S′. Place a pencil point P vertically against the thread in the angle SPS′, and move the pencil all round so as to keep the two portions of the thread between it and the fixed ends always tight. Then the pencil point describes an ellipse.



(For, SP+S′P=length of thread=constant.)

Note. Gardener's method. This is popularly called the **gardener's method** of describing an ellipse.]

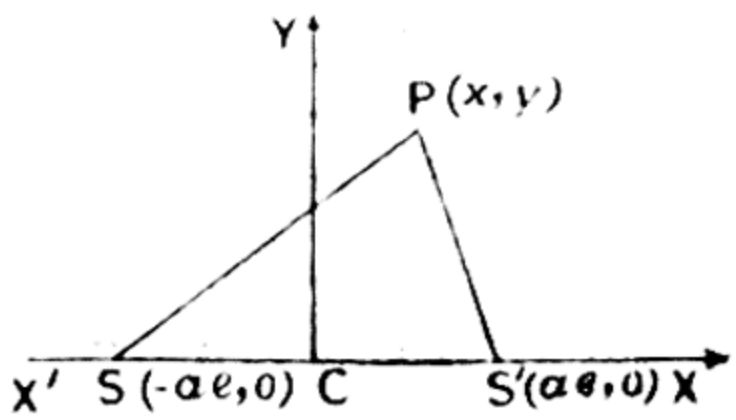
2. An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is constant.

From the above definition of the ellipse, deduce its equation in the standard form. [*A.U. 1940*]

[**Sol.** Let S, S′ be the fixed pts., and $2a$ the constant sum of the distances.

Let $SS' = 2ae$.

Take C, the mid-pt. of SS′, as origin, CS′ as the x-axis and CY ⊥ to it, as the y-axis. Then the co-ordinates of S, S′ are $(-ae, 0)$, $(ae, 0)$.



(*i*) Let P (x, y) be *any* pt. on the ellipse.

(*ii*) Then $SP+S'P=2a$.

(*iii*) $\therefore\ \sqrt{(x+ae)^2+y^2}+\sqrt{(x-ae)^2+y^2}=2a$ ...(1)

Now identically,

$$[\,(x+ae)^2+y^2\,]-[\,(x-ae)^2+y^2\,] =4aex \quad ...(2)$$

Dividing (2) by (1),

$$\sqrt{(x+ae)^2+y^2}-\sqrt{(x-ae)^2+y^2} =2ex \quad ...(3)$$

$\frac{4aex}{2a}=2ex$

Adding (1) and (3), $2\sqrt{(x+ae)^2+y^2}=2a+2ex$

or $\sqrt{(x+ae)^2+y^2}=a+ex$

or, squaring, $x^2+2aex+a^2e^2+y^2=a^2+2aex+e^2x^2$

or $x^2(1-e^2)+y^2=a^2(1-e^2)$

or, dividing thro' out by $a^2(1-e^2)$*, $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$

or, *putting* $a^2(1-e^2)=b^2$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is the required equation.

Note. Cf. Solution of Ex. 5, Art. 12.]

3. The focal distance of an end of the minor axis of an ellipse is k and the distance between its foci $2h$. Find the lengths of its semi-axes.

[*A.U. 1942*]

137. (*b*) **If the tangent at any point P of an ellipse meets the major axis in T, and PN is perpendicular on the major axis, then $CT.CN=CA^2$.**

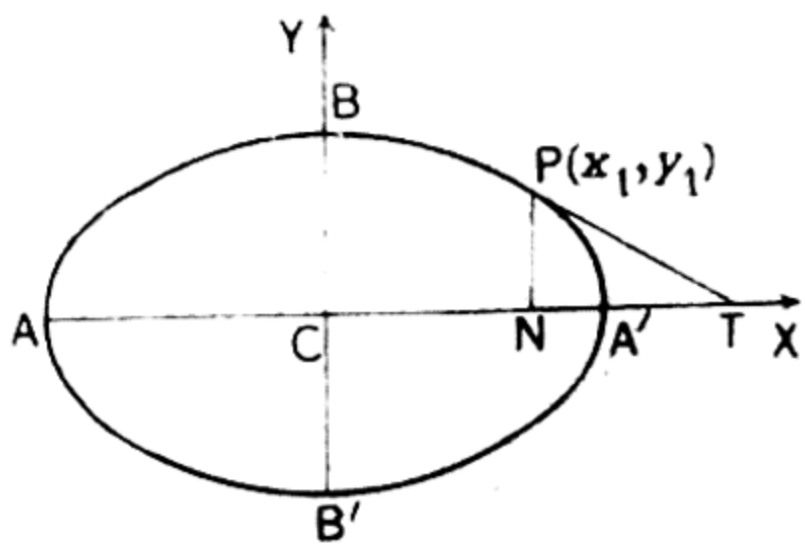
Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P be the pt. (x_1, y_1).

The equation of the tangent at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad ...(1)$$

It meets the major axis where,

putting $y=0$ in (1), $\frac{xx_1}{a^2}=1$



$\therefore\ xx_1=a^2$, *i.e.*, $CT.CN=CA^2$.

Cor. To find the length of the subtangent.

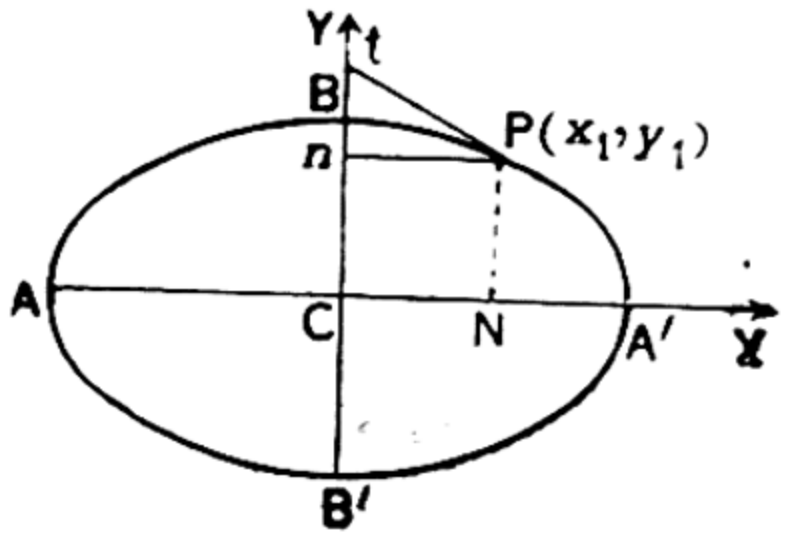
$$\text{Subtangent} = NT=CT-CN=\frac{a^2}{x_1}-x_1 \left[\because\ CT=\frac{a^2}{x_1} \text{ (Art. 137,(}b\text{))}\right]$$

$$=\frac{a^2-x_1^2}{x_1}.$$

137. (*c*) **If the tangent at any point P of an ellipse meets the minor axis in t, and Pn is perpendicular on the minor axis, then $Ct.Cn=CB^2$.**

***Why this step.** See the footnote * on p. 185.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P be the pt. (x_1, y_1).



The equation of the tangent at P is

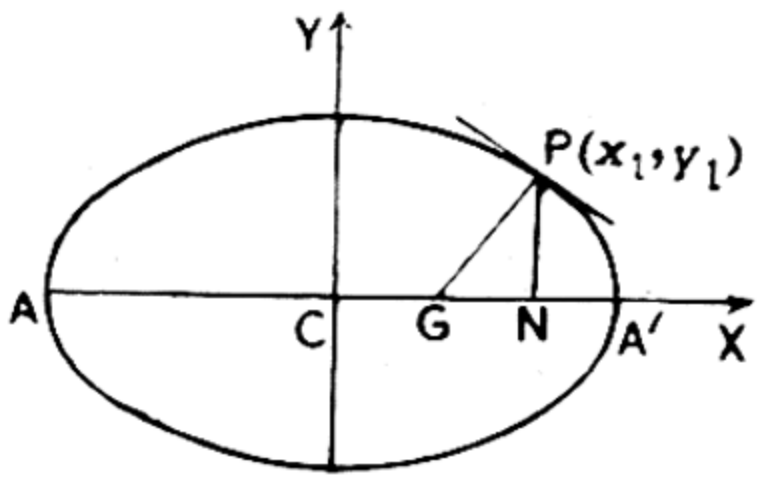
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \ldots(1)$$

It meets the minor axis where, putting $x=0$ in (1), $\frac{yy_1}{b^2}=1$,

$\therefore\ yy_1=b^2$, *i.e.*, $Ct.Cn=CB^2$. [$\because\ Cn=NP=y_1$]

137. (*d*) **If the normal at any point P of an ellipse meets the major axis in G, and PN is perpendicular on the major axis, then $CG=e^2.CN$.***

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P be the pt. (x_1, y_1).



The equation of the normal at P is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2 \quad \ldots(1)$$

It meets the major axis where, putting $y=0$ in (1), $\frac{a^2x}{x_1}=a^2-b^2$

$$\therefore\ x=\frac{a^2-b^2}{a^2}\,x_1=\frac{a^2-a^2(1-e^2)\dagger}{a^2}\,x_1=e^2x_1, \textit{ i.e.}, CG=e^2.CN.$$

[**N.B.** *If the normal at any point $P(x_1, y_1)$ of an ellipse meets the major axis in G, then* $\mathbf{CG=e^2.x_1}$.]

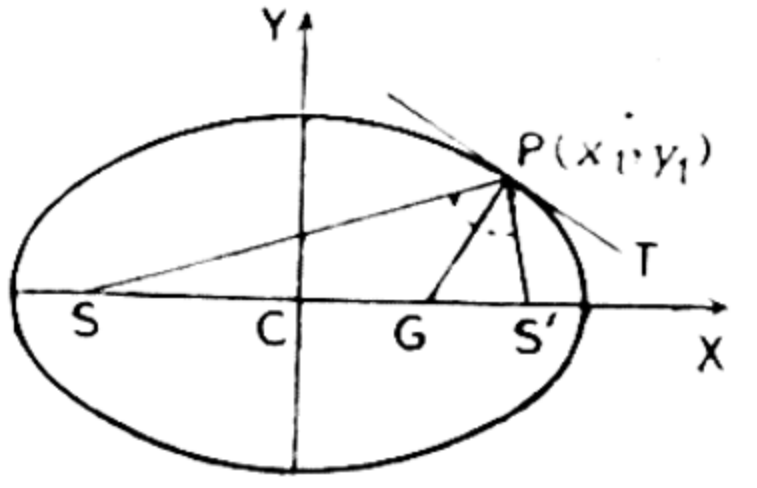
Cor. To find the length of the subnormal.

Subnormal $=NG=CN-CG=x_1-e^2x_1$ [$\because\ CG=e^2x_1$ (Art. 137, (*d*))]
$=x_1(1-e^2)$.

137. (*e*) **The normal at any point of an ellipse bisects the angle between the focal distances of the point.‡**

Let S, S′ be the foci of the ellipse.

Let the normal at $P(x_1, y_1)$ meet the major axis in G.



Then $\frac{SG}{GS'}\S=\frac{CS+CG}{CS'-CG}$

$=\frac{ae+e^2x_1}{ae-e^2x_1}$

[$\because\ CS=ae,\ CG=e^2x_1,\ CS'=CS=ae$]

$=\frac{e(a+ex_1)}{e(a-ex_1)}$

---

*[*P. U. 1955*]

†**Why this step.** $\because$ the result to be proved involves e^2 and not b^2,
$\therefore$ we change b^2 into $a^2(1-e^2)$.

‡[*P(P).U. 1951 S*]

§**Why this step.** In order to prove that PG bisects $\angle SPS'$, we have to prove that $\frac{SG}{GS'}=\frac{SP}{S'P}$, $\therefore$ we consider $\frac{SG}{GS'}$.

$$=\frac{a+ex_1}{a-ex_1}=\frac{SP}{S'P}$$ [Art. 137, (*a*), Cor.]

$\therefore$ PG bisects $\angle SPS'$. [From Geometry]

Cor. The tangent at any point of an ellipse bisects the exterior angle between the focal distances of the point.*

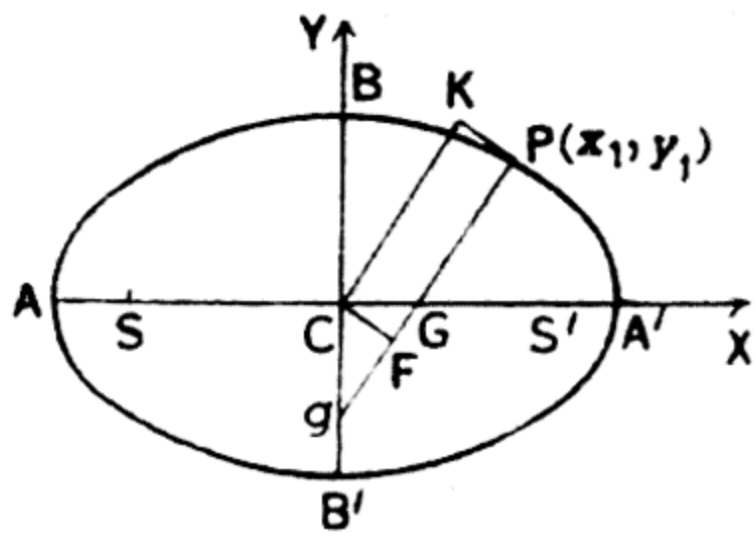
Proof. PG bisects the interior angle between SP and S′P. [Art. 137, (*e*)]

But PT is $\perp$ to PG, $\therefore$ PT bisects the exterior angle between SP and S′P.

EXAMPLE

****If the normal at any point P of an ellipse with centre C and foci S, S′ meets the major and minor axes in G and g, and if CF be the perpendicular upon the normal, prove that PF.PG=b², PF.Pg=a², where a is the semi-axis major and b the semi-axis minor.** [*A.U.*]

[Sol. (*i*) Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P be the pt. (x_1, y_1). The equation of the tangent at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ...(1)



From C draw CK $\perp$ on the tangent.

Then PF=CK

$=\perp$ from C(0, 0) on the tangent (1)

$$=\frac{1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}}}=\frac{a^2b^2}{\sqrt{b^4x_1^2+a^4y_1^2}} \quad ...(2)$$

Now the equation of the normal at P is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ ...(3)

It meets the major axis where, putting $y=0$ in (3), $\frac{a^2x}{x_1}=a^2-b^2$.

$$\therefore \quad x=\frac{a^2-b^2}{a^2}x_1, \qquad \therefore \text{ G is } \left(\frac{a^2-b^2}{a^2}x_1,\ 0\right).$$

$$\therefore \text{ PG}=\sqrt{\left(x_1-\frac{a^2-b^2}{a^2}x_1\right)^2+(y_1-0)^2}$$

$$=\sqrt{\frac{b^4}{a^4}x_1^2+y_1^2}=\frac{\sqrt{b^4x_1^2+a^4y_1^2}}{a^2} \quad ...(4)$$

$$\therefore \text{ from (2) and (4), PF.PG}=\frac{a^2b^2}{\sqrt{b^4x_1^2+a^4y_1^2}}\cdot\frac{\sqrt{b^4x_1^2+a^4y_1^2}}{a^2}=b^2.$$

---

*[*P(P). U. 1951S*]

(*ii*) The normal at P meets the minor axis where, putting $x=0$ in (3),

$$-\frac{b^2y}{y_1}=a^2-b^2, \quad \therefore\ y=\frac{b^2-a^2}{b^2}y_1, \quad \therefore\ g \text{ is } \left(0, \frac{b^2-a^2}{b^2}y_1\right).$$

$$\therefore \quad Pg=\sqrt{(x_1-0)^2+\left(y_1-\frac{b^2-a^2}{b^2}y_1\right)^2}$$

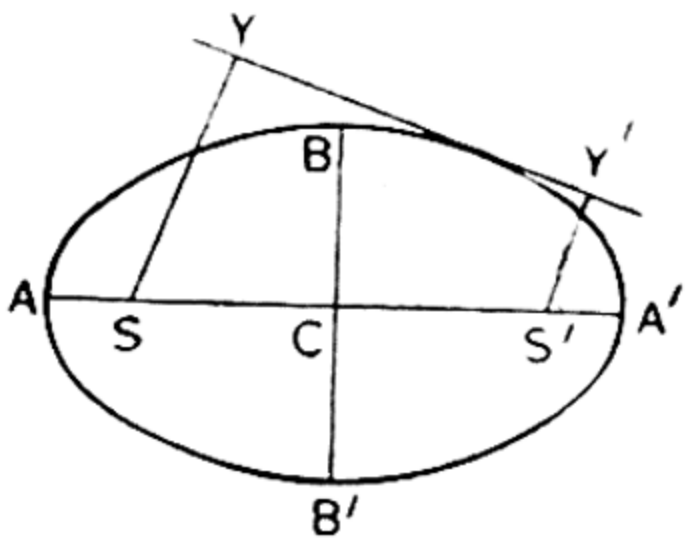
$$=\sqrt{x_1^2+\frac{a^4y_1^2}{b^4}}=\frac{\sqrt{b^4x_1^2+a^4y_1^2}}{b^2} \qquad \ldots(5)$$

$$\therefore \text{ from (2) and (5), } PF.Pg=\frac{a^2b^2}{\sqrt{b^4x_1^2+a^4y_1^2}}\cdot\frac{\sqrt{b^4x_1^2+a^4y_1^2}}{b^2}=a^2.$$

137. (*f*) **The locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle*, and the product of the perpendiculars is equal to the square on the semi-minor axis.†**

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



(*i*) The equation of *any* tangent to the ellipse, in the *m*-form, is

$$y=mx+\sqrt{a^2m^2+b^2} \qquad \ldots(1)$$

The slope of any line $\perp$ to it $=-\frac{1}{m}$

[−ve reciprocal]

$\therefore$ the equation of the $\perp$ SY from S$(-ae, 0)$ on the tangent (1) is

$$y-0=-\frac{1}{m}(x+ae). \qquad \ldots(2)$$

[To find the locus of Y, the foot of the $\perp$ (*i.e.*, pt. of intersection of (1) and (2)), *i.e.*, to eliminate *m* from (1) and (2).

(Misc. Ex. 23, Chap. III)]

Now (1) and (2) may be written as

[Constant terms on R.H.S. (*Note these steps*)]

$$y-mx=\sqrt{a^2m^2+b^2} \qquad \ldots(3$$

and

$$x+my=-ae \qquad \ldots(4)$$

Squaring (3) and (4),

$$y^2-2mxy+m^2x^2=a^2m^2+b^2$$

and

$$x^2+2mxy+m^2y^2=a^2e^2.$$

Adding, $x^2(1+m^2)+y^2(1+m^2)=a^2m^2+b^2+a^2e^2$

[But $b^2=a^2(1-e^2)$‡]

$$=a^2m^2+a^2(1-e^2)+a^2e^2$$

$$=a^2(1+m^2).$$

Dividing thro'out by $1+m^2$,

$$x^2+y^2=a^2,$$ which is the required locus.

This is the auxiliary circle. [Art. 134, Cor.]

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*[*P.U. 1957S*] †[*P(P). U. 1955*]

‡**Why this step.** In order to prove that the locus is the auxiliary circle $x^2+y^2=a^2$ (not b^2), we change b^2 into $a^2(1-e^2)$.

Similarly the locus of Y′, the foot of the ⊥ from the other focus S′ on the tangent, is the auxiliary circle.

(*ii*) **To prove that SY.S′Y′=CB².**

The equation of the tangent is $y=mx+\sqrt{a^2m^2+b^2}$

or $$mx-y+\sqrt{a^2m^2+b^2}=0 \quad ...(1)$$

$$\therefore \text{ SY} = \perp \text{ from } (-ae, 0) \text{ on } (1)=\frac{-aem+\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}},$$

$$\text{S′Y′}= \perp \text{ from } (ae, 0) \text{ on } (1)=\frac{aem+\sqrt{a^2m^2+b^2}}{\sqrt{m^2+1}}.$$

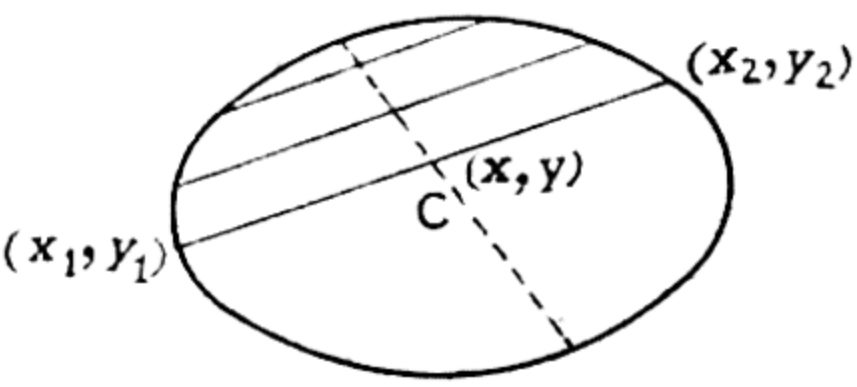
$$\therefore \text{ SY.S′Y′}=\frac{a^2m^2+b^2-a^2e^2m^2}{m^2+1}=\frac{m^2a^2(1-e^2)+b^2}{m^2+1}$$

[But $a^2(1-e^2)=b^{2*}$]

$$=\frac{m^2b^2+b^2}{m^2+1}=\frac{b^2(m^2+1)}{m^2+1}=b^2=\text{CB}^2.$$

137. (*g*) **Diameter property. The locus of the middle points of a system of parallel chords of an ellipse is a straight line passing through the centre.**†

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$



(*i*) Let (x_1, y_1), (x_2, y_2) be the ends of *any* chord of the system and (x, y) its mid-pt.

Then $x=\frac{x_1+x_2}{2}$, $y=\frac{y_1+y_2}{2}$ ...(1)

Let m be the slope of the ∥ chords.

Then $$m=\frac{y_2-y_1}{x_2-x_1} \quad ...(2)$$

(*ii*) [To find the value of $\frac{y_2-y_1}{x_2-x_1}$.]

∵ (x_1, y_1), (x_2, y_2) lie on the ellipse,

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \quad ...(3)$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1 \quad ...(4)$$

Subtracting (3) from (4), $\frac{x_2^2-x_1^2}{a^2}+\frac{y_2^2-y_1^2}{b^2}=0.$

Factorising, $\frac{(x_2-x_1)(x_2+x_1)}{a^2}+\frac{(y_2-y_1)(y_2+y_1)}{b^2}=0,$

***Why this step.** In order to prove that SY.S′Y′=CB²=b^2 (not a^2), we change $a^2(1-e^2)$ into b^2.

†[*P.U. 1956*]

or $$\frac{(y_2-y_1)(y_2+y_1)}{b^2} = -\frac{(x_2-x_1)(x_2+x_1)}{a^2}$$

$$\therefore \quad \frac{y_2-y_1}{x_2-x_1} = -\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1} \quad \ldots(5)$$ [Cf. Step (*ii*), Art. 122.]

(*iii*) Substituting the values of $\frac{y_2-y_1}{x_2-x_1}$ $(=m)$, $x_2+x_1(=2x)$, $y_2+y_1(=2y)$ from (2) and (1) in (5),

$$m=-\frac{b^2}{a^2}\cdot\frac{2x}{2y}$$ [Cancel 2]

or $$\mathbf{y=-\frac{b^2}{a^2m}x}$$, which is the required locus.

This is a st. line passing thro' the origin, *i.e.*, thro' the centre of the ellipse.

[**N.B.** *The equation of the diameter (Def., Art. 84) of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *bisecting parallel chords of slope m, is* $\mathbf{y=-\frac{b^2}{a^2m}x}$.]

[**Aid to mémory.** Putting $b=a$, the ellipse becomes the circle $x^2+y^2=a^2$ (Art. 119, (*a*), Note 4). Hence the *equation of the diameter of the circle* $x^2+y^2=a^2$, *bisecting parallel chords of slope m, is*

$$\mathbf{y=-\frac{1}{m}x}.$$ (Cf. Art. 84.)]

[**Rule to find the locus of the middle points of a system of parallel chords of an ellipse :** *See Rule of Art. 84.*]

EXAMPLES

1. Show that the tangents at the extremities of a diameter of an ellipse are parallel. [*P(P). U. 1950S*]

[**Hint.** Let (x_1, y_1) be an extremity of a diameter. Then the other extremity is $(-x_1, -y_1)$ ($\because$ the mid-pt. of a diameter is the centre (0,0)).]

2. Find the locus of the middle points of the chords of the ellipse $2x^2+3y^2=4$, each of which makes an angle of 45° with the x-axis. [*P. U. 1950*]

****3.** (*a*) **Equation of a chord in terms of its mid-point. The equation of the chord of the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, **whose mid-point is** (x_1, y_1), **is** $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$. [*P. U. 1957*]

[**N.B.** *The equation of the chord of the ellipse* $S=0$, *whose mid-point is* (x_1, y_1), *is* $\mathbf{T=S_1}$, *where* $T=0$ *is the equation of the tangent at* (x_1, y_1), *and* S_1 *is the result of substituting the co-ordinates of the point* (x_1, y_1) *in S.*]

****(*b*) If chords of an ellipse pass through a fixed point, their mid-points lie on another ellipse.** [*P. U. 1957*]

[**Hint.** (*a*) Proceed as in Method I, solved Ex. 1, Art. 84. (Here the mid-pt. (x_1,y_1) lies on the diameter $y=-\frac{b^2}{a^2m}x$ (Art. 137, (*g*)).

Sol. (*b*) The equation of the chord, whose mid-pt. is (x_1, y_1), is

$(T=S_1, i.e.,) \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

or $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$.

If it passes thro' the fixed pt. (h, k), then

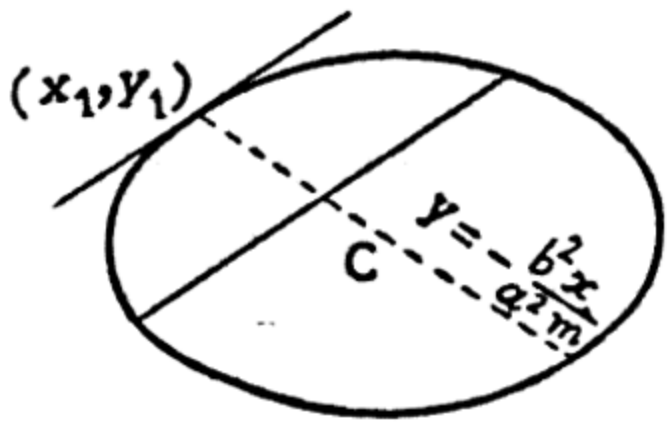
$\frac{hx_1}{a^2}+\frac{ky_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$.

$\therefore$ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y)]

$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$, which is an ellipse.]

4. The tangent at a vertex of a diameter of an ellipse is parallel to the chords bisected by the diameter.

[**Sol.** Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



(*i*) Let m be the slope of the ‖ chords.

Then the equation of the diameter bisecting them is $y=-\frac{b^2}{a^2m}x$ ...(1) [Art. 137,(*g*)]

Let (x_1, y_1) be a vertex of the diameter.

$\because$ (x_1, y_1) lies on (1)

$\therefore$ $y_1=-\frac{b^2}{a^2m}x_1$ ...(2)

(*ii*) The equation of the tangent at the vertex is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(*iii*) $\therefore$ the slope of the tangent $=-\frac{\frac{x_1}{a^2}}{\frac{y_1}{b^2}}$ $\left[-\frac{\text{coeff. of } x}{\text{coeff. of } y}\right]$

$=-\frac{b^2x_1}{a^2y_1}$

[Substitute the value of y_1 from (2) (*Note this step*)]

$=-\frac{b^2x_1}{a^2\left(-\frac{b^2x_1}{a^2m}\right)}=m=$the slope of the chords

$\therefore$ the tangent is ‖ to the chords.]

5. If one diameter of an ellipse bisects chords parallel to a second, the second diameter will bisect all chords parallel to the first.

[**Sol.** Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

and the equations of the diameters be

$y=mx$ ...(1)

and $y=m'x$ ...(2)

The equation of the diameter bisecting chords ∥ to the diameter (2) (*i.e.*, of slope m') is

$$y=-\frac{b^2}{a^2m'}x \quad ...(3) \qquad [y=-\frac{b^2}{a^2m}x]$$

If the diameter (1) bisects chords ∥ to the diameter (2), then (1) is the same as (3).

$\therefore$ comparing coefficients, $m=-\frac{b^2}{a^2m'}$ or $\mathbf{mm'=-\frac{b^2}{a^2}}$.

The *symmetry** of this result shows that the diameter (2) bisects chords ∥ to the diameter (1).

Conjugate diameters. Def.† Two diameters of an ellipse are said to be **conjugate** if each bisects chords parallel to the other.

****Cor. 1. Condition for conjugate diameters.†** *The condition, that* $y=mx$, $y=m'x$ *may be conjugate diameters of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *is*

$$\mathbf{mm'=-\frac{b^2}{a^2}}.$$

****Cor. 2. Equation of conjugate diameter.** *The equation of the diameter conjugate to the diameter* $y=mx$ *of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is*

$$\mathbf{y=-\frac{b^2}{a^2m}x.}]$$

****6.** Show that the diameters $y+x=0$ and $2y-x=0$, are conjugate diameters of the ellipse $x^2+2y^2=3$.

****7.** In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, write down the equation to the diameter which is conjugate to the diameter

(*i*) $x+y=0$, (*ii*) $y=\frac{b}{a}x$.

****8.** Find the eccentricity of the ellipse if $y=x$ and $3y=-2x$ are a pair of its conjugate diameters. [*M. U. 1942S*]

MISCELLANEOUS EXAMPLES ON CHAPTER VII

1. Find the equation of the ellipse which has the point $(-1, 1)$ for focus, the line $3x-4y=0$ for directrix, and whose eccentricity is $\frac{5}{6}$.

2. Find the lengths of the axes and the latera recta of the ellipse

$$4x^2+3y^2=24.$$ [*P. U. 1930S*]

Find also the co-ordinates of the vertices and the foci, and the equations of the directrices. [*D. U. 1932*]

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*"*The symmetry of this result shows*" means that "on changing m to m' and m' to m, the result remains unchanged."

For, $mm'=-\frac{b^2}{a^2}$ becomes $m'm=-\frac{b^2}{a^2}$.

†[*P. U. 1953S*]

3. What does $2x^2+3y^2=6$ represent ? What are the co-ordinates of the ends of the latera recta ? [*P. U.*]

4. Show that $4x^2+y^2-8x+2y+1=0$ represents an ellipse. Find its eccentricity, co-ordinates of its foci, and the equations to the directrices. [*P. U. 1933*]

[**Sol.** The equation is $4x^2+y^2-8x+2y+1=0$.

Transposing, $4(x^2-2x)+(y^2+2y)=-1$.

[Constant term on R.H.S.]

Completing the squares,

$$4(x^2-2x+1)+(y^2+2y+1)=-1+4+1.$$

[Adding 4+1 to both sides]

or $$4(x-1)^2+(y+1)^2=4.$$

Put $x-1=x'$, $y+1=y'$.

Then $4x'^2+y'^2=4$ or $\dfrac{x'^2}{1}+\dfrac{y'^2}{4}=1$ ...(1)

which is an ellipse. Here denom. of y'^2 > denom. of x'^2

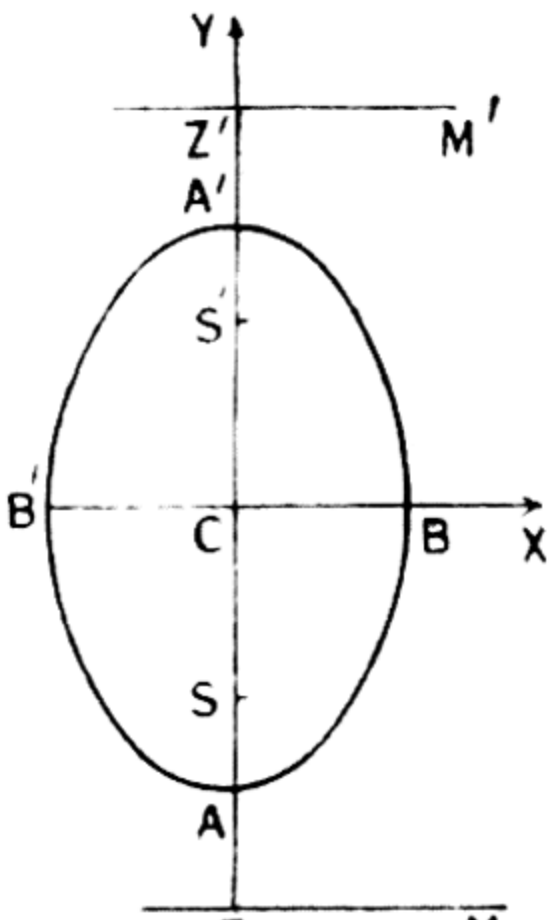
[*i.e., second standard form*]

$\therefore$ the major axis lies along the new y-axis.

From (1), (greater denominator) $a^2=4$, and (lesser denominator) $b^2=1$.

(*i*) $b^2=a^2(1-e^2)$, $\therefore$ $1=4(1-e^2)=4-4e^2$,

or $4e^2=3$, $\therefore$ $e=\dfrac{\sqrt{3}}{2}$.



(*ii*) The foci lie on the *major* axis, *i.e.*, on the y-axis on opposite sides of C such that

$$CS=CS'=ae=2.\frac{\sqrt{3}}{2}=\sqrt{3}$$

$\therefore$ the foci are $x'=0$, $y'=\pm\sqrt{3}$

[$(0, \pm ae)$]

or $x-1=0$, $y+1=\pm\sqrt{3}$ or $x=1$, $y=-1\pm\sqrt{3}$,

$\therefore$ the foci are $(1, -1\pm\sqrt{3})$.

(*iii*) The directrices ZM, Z'M' are $\perp$ to the major axis, *i.e.*, $\perp$ to the y-axis or $\parallel$ to the x-axis on opposite sides of C such that

$$CZ=CZ'=\frac{a}{e}=\frac{2}{\frac{\sqrt{3}}{2}}=\frac{4}{\sqrt{3}}.$$

$\therefore$ the equations of the directrices are $\left[\ y=\pm\dfrac{a}{e}\ \right]$

$$y'=\pm\frac{4}{\sqrt{3}} \quad \text{or} \quad y+1=\pm\frac{4}{\sqrt{3}} \quad \text{or} \quad \sqrt{3}y+\sqrt{3}\pm 4=0.\ ]$$

5. Find the eccentricity, the foci and directrices of the ellipse

(*i*) $3x^2+4y^2-12x-8y+4=0$. [*B.U.*]

(*ii*) $8(x+1)^2+6(y-1)^2-1=0$.

6. Show that the equation $2x^2+y^2=3x$ represents an ellipse, find its eccentricity, latus rectum and the co-ordinates of the foci.

**Show also that the origin is at an extremity of the minor axis.

7. Determine the centre, equations of the major and minor axes, their lengths, the foci and the directrices of the ellipse

$$25x^2+9y^2-150x-90y+225=0.$$ [*B.U. 1939*]

8. Show that the angle between the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=ab$ at their points of intersection is $\tan^{-1}\frac{a-b}{\sqrt{ab}}$. [*A.U.*]

9. *Find the co-ordinates of the point on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at which the tangent makes equal angles with the axes.* [*P(P). U. 1956S*]

Prove also that the length of the perpendicular from the centre on the tangent is $\sqrt{\frac{a^2+b^2}{2}}$.

[**Hint.** (*i*) Let (x_1, y_1) be the required pt. The equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

$\because$ it makes equal angles with the axes,
$\therefore$ its slope $=\pm 1$, (Art. 20, Ex. 3)

i.e., $-\frac{b^2x_1}{a^2y_1}=\pm 1$ ... (1)

Also $\because$ (x_1, y_1) lies on the ellipse, $\therefore$ $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ ... (2)

Substitute the value of y_1 from (1) in (2), and find the values of x_1.

Substitute these values of x_1 one by one in (1), and find the corresponding values of y_1.]

10. *Find the points on the ellipse* $9x^2+16y^2=144$ *such that the tangents are equally inclined to the axes. Prove that the length of the perpendicular from the centre on either tangent is* $\frac{5}{2}\sqrt{2}$. [*D. U. 1933*]

11. Find the co-ordinates of the point on the ellipse $x^2+3y^2=37$ at which the normal is parallel to the line $6x-5y=2$. [*P.U. 1955*]

12. If the normal at the end of a latus rectum of an ellipse pass through one extremity of the minor axis, show that the eccentricity of the curve is given by the equation $e^4+e^2-1=0$. [*P.U. 1951*]

[**Hint.** Draw a rough Fig. Find the equation of the normal at $\left(-ae, \frac{b^2}{a}\right)$.

If it passes thro' $(0, -b)$, then it will be found that

$$ab=a^2-b^2 \quad [\text{ But } b^2=a^2(1-e^2)]$$

or *a*. $a\sqrt{1-e^2}=a^2-a^2(1-e^2)=a^2e^2$.

(Cancel a^2) or $\sqrt{1-e^2}=e^2$. Square and transpose.]

13. Find the length of the line $x=y$ intercepted by the ellipse $2x^2+3y^2=10$.

14. Show that the line $y\sqrt{3}=x+3$ touches both the parabola $y^2=4x$ and the ellipse $2x^2+3y^2=6$ and find the points of contact. [*B. U. 1931*]

15. Prove that the polar of the point $(-3, 4)$ with respect to the parabola $y^2=4x$ touches the ellipse $x^2+2y^2=3$ and find the co-ordinates of the point of contact. [*P.U. 1949*]

16. Common tangents to two ellipses. *Find a common tangent to the ellipses* $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ *and* $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$. [*P.U. 1932*]

[**Hint.** Let m be the slope of a tangent to the first ellipse.
Then its equation is $y=mx\pm\sqrt{(a^2+b^2)m^2+b^2}$ ... (1)
$(y=mx\pm\sqrt{a^2m^2+b^2})$
If it touches the second ellipse, then
$\pm\sqrt{(a^2+b^2)m^2+b^2}=\pm\sqrt{a^2m^2+(a^2+b^2)}$ ... (2) $[c=\pm\sqrt{a^2m^2+b^2}]$
Square (2), find the values of m, and substitute them one by one in (1).]

17. Trace the conics $\frac{x^2}{9}+\frac{y^2}{16}=1$, and $\frac{x^2}{16}+\frac{y^2}{9}=1$. Prove that the points of intersection form the four corners of a square.
Obtain the equation of a common tangent to both. [*P(P).U. 1956*]

18. A circle of radius r is concentric with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that the common tangent is inclined to the major axis at an angle

$$\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}.$$

19. Prove that the equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be written in the form $x\cos\alpha+y\sin\alpha=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$.
Hence find the locus of the intersection of perpendicular tangents.
[*Pesh. U. 1951*]

[**Hint.** Find the condition that $x\cos\alpha+y\sin\alpha=p$ ...(1) may touch the ellipse (Art. 126,Ex. 5). It will be found that $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$ or $p=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$, $\therefore$ from (1), the equation of the tangent is $x\cos\alpha+y\sin\alpha=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$ ...(2)
Changing α to $90°+\alpha$, the equation of a $\perp$ tangent is $-x\sin\alpha+y\cos\alpha=\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}$ ... (3). Square (2) and (3), and add.]

20. Tangents are drawn from the point (2, 3) to the ellipse $9x^2+16y^2=144$. Find their equations and the equation of their chord of contact.

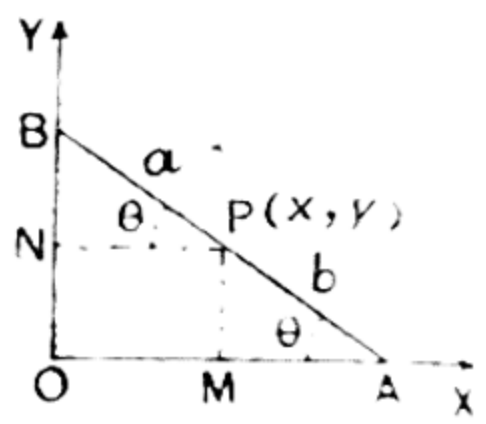
21. Tangents TP, TQ are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, centre C. Prove that the product of the slopes of PQ and TC is constant.
[*P. U. 1934 S*]

22. Find with respect to the ellipse $4x^2+5y^2=20$,
(*i*) the polar of the point $(-\frac{1}{2}, \frac{3}{5})$, and

(*ii*) the pole of the line $2x-5y+10=0$.

23. *A straight line of given length slides with its ends on two lines a right angles to each other. Find the locus of any point on the line.* [*M. U.*]

[**Sol.** Let the line AB slide with its ends on the two $\perp$ lines OA, OB. Take OA, OB as axes (Art. 8-I, Rule I). Let P(x,y) be any pt. on AB, BP$=a$, PA$=b$, $\angle BAO=\theta$.



From P draw PM and PN $\perp$s on the x-axis and y-axis.

Then $\angle BPN=\angle BAO$ (Corresponding $\angle$s)
$=\theta$.

Now $x=OM=NP$ $\left[\text{But } \frac{NP}{PB}=\cos\theta, \quad \therefore\ NP=PB\cos\theta\right]$
$=a\cos\theta$.

Again $y=MP$ $\left[\text{But } \frac{MP}{AP}=\sin\theta, \quad \therefore\ MP=AP\sin\theta\right]$
$=b\sin\theta$

$\therefore \quad \cos\theta=\frac{x}{a}$, $\sin\theta=\frac{y}{b}$. Squaring and adding, $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

[$\because \quad \cos^2\theta+\sin^2\theta=1$]

$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is the required locus. This is an ellipse.]

24. A line of given length $2a$ moves so that its ends always lie on two fixed perpendicular straight lines. Prove that the locus of its middle point is a circle. [*R. B. 1932*]

25. If ϕ_1, ϕ_2 be the eccentric angles of the extremities of a focal chord of an ellipse of eccentricity e, prove that

$$\cos\frac{\phi_1-\phi_2}{2}=e\cos\frac{\phi_1+\phi_2}{2}.$$ [*M.U. 1931*]

26. (*a*) Prove that the locus of the middle points of the portions of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ included between the axes is the curve

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4.$$ [*P. U. 1954*]

(*b*) Find the locus of the mid-point of the normal PG at P, G being on the major axis.

27. If a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose centre is C, meets the major axis in T and the minor axis in t, prove that

$$\frac{a^2}{CT^2}+\frac{b^2}{Ct^2}=1.$$ [*P. U. 1935*]

28. The normal at any point P of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ cuts the major and minor axes at G and g respectively ; prove that $9CG^2+4Cg^2=25$, where C is the centre of the ellipse. [*A.U. 1948*]

29. Find the condition that the line $x\cos\alpha+y\sin\alpha=p$ may be a normal to the ellipse $2x^2+3y^2=1$. [*J. & K. U. 1949*]

30. Given the ellipse $36x^2+100y^2=3600$, find the equations and the lengths of the focal radii drawn through the point $(8,\frac{18}{5})$. [*D.U.*]

31. *Prove that the normal* [*at the point* (x_1, y_1) *of an ellipse*] *divides the major axis into segments whose product is* $a^2-e^4x_1^2$. [*P. U.*]

[Sol. In the Fig. of Art. 137, (*d*), $AG.A'G=(CA+CG)(CA'-CG)$
$=(a+e^2x_1)(a-e^2x_1)$ [$\because$ $CG=e^2x_1$ (Art. 137, (*d*))]
$=a^2-e^4x_1^2$.]

32. *Prove that the sum of the squares of the perpendiculars on any tangent to an ellipse from two points on the minor axis, each distant* $\sqrt{a^2-b^2}$ *from the centre is* $2a^2$. [*C. U. 1953*]

33. Find the locus of the point of intersection of the two straight lines $\frac{tx}{a}-\frac{y}{b}+t=0$ and $\frac{x}{a}+\frac{ty}{b}-1=0$.

34. In an ellipse the length of the subtangent corresponding to the point $(3,\frac{12}{5})$ is $\frac{16}{3}$. Prove that its eccentricity is $\frac{4}{5}$. [*A.U. 1939*]

35. Prove that in an ellipse the pole of a chord lies on the diameter which bisects the chord. [*P. U. 1953 S*]

[**Hint.** Let the equation of the chord be $y=mx+c$ ...(1). Find its pole w. r. t. the ellipse. Show that it lies on the diameter bisecting the chord (1), *i.e.*, on $y=-\frac{b^2}{a^2m}x$.]

****36.** *Prove that the polar of any point on a diameter of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is parallel to the conjugate diameter.* [*P.U. 1945*]

[**Sol.** Let the equation of the diameter be $y=mx$, and (x_1, y_1) any pt. on it so that $y_1=mx_1$ ...(1)

The equation of the polar of (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

$\therefore$ the slope of the polar $=-\frac{\frac{x_1}{a^2}}{\frac{y_1}{b^2}}=-\frac{b^2x_1}{a^2y_1}$ [Substitute the value of y_1 from (1)]

$=-\frac{b^2x_1}{a^2mx_1}=-\frac{b^2}{a^2m}$

= the slope of the conjugate diameter $\left[y=-\frac{b^2}{a^2m}x\right]$

$\therefore$ the polar is $\parallel$ to the conjugate diameter.]

11. The tangent at a vertex of a diameter of a hyperbola is parallel to the chords bisected by the diameter.

[**Hint.** Proceed as in Ex. 4, Art. 137, (*g*).]

12. If one diameter of a hyperbola bisects chords parallel to a second, the second diameter will bisect all chords parallel to the first.

[**Hint.** Proceed as in Ex. 5, Art. 137, (*g*).]

[**Conjugate diameters. Def.** Two diameters of a hyperbola are said to be **conjugate** if each bisects chords parallel to the other.

****Cor. 1. Condition for conjugate diameters.** *The condition, that* $y=mx$, $y=m'x$ *may be conjugate diameters of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *is*

$$\mathbf{mm'=\frac{b^2}{a^2}}.$$

****Cor. 2. Equation of conjugate diameter.** *The equation of the diameter conjugate to the diameter* $y=mx$ *of the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *is*

$$\mathbf{y=\frac{b^2}{a^2m}x}.\ ]$$

MISCELLANEOUS EXAMPLES ON CHAPTER VIII

1. Prove that the straight lines $\frac{x}{a}+\frac{y}{b}=m$ and $\frac{x}{a}-\frac{y}{b}=\frac{1}{m}$ always meet on a hyperbola.

2. Prove that, when m varies, the point of intersection of the straight lines $\sqrt{3}x-y-4\sqrt{3}m=0$ and $\sqrt{3}mx+my-4\sqrt{3}=0$ describes a hyperbola whose eccentricity is 2. [*B.U. 1943*]

3. Find the equation to the hyperbola of given transverse axis whose vertex bisects the distance between the centre and the focus.

4. Find the equation to the hyperbola whose eccentricity is $\frac{5}{4}$, whose focus is (a, 0), and whose directrix is $4x+3y=a$. [*P.U. 1956*]

**Find also the co-ordinates of the centre and the equation to the other directrix.

[**Hint.** (*ii*) Find Z, the foot of the directrix, *i.e.*, the pt. of intersection of the directrix and the $\perp$ from the focus on the directrix. It will be found that Z is $\left(\frac{13}{25}a, -\frac{9}{25}a\right)$. Now $\frac{CZ}{CS}=\frac{\frac{a}{e}}{ae}=\frac{1}{e^2}$, $\therefore$ C divides ZS *externally* in the ratio $1 : e^2$, *i.e.*, 16 : 25. Find the co-ordinates (x, y) of Z′ such that C is the mid-pt. of ZZ′. It will be found that Z′ is $\left(-\frac{89}{75}a, -\frac{41}{25}a\right)$. The other directrix is $\parallel$ to the given directrix thro' Z′.]

5. (*a*) Find the lengths of the axes, and the eccentricity of the hyperbola $x^2-2y^2+2x-3=0$.

(*b*) Find the equations of the axes of the hyperbola

$$16x^2-y^2+64x+4y+44=0.$$ [*B.U. 1945*]

6. Find the centre, eccentricity, foci and directrices of the hyperbola $9x^2-16y^2+18x+32y-151=0$.

7. Find the centre and the length of the transverse axis of the hyperbola $x^2-2y^2+2x-8y=1$.

8. Find the co-ordinates of the point at which the line $3x+4y=7$ is a normal to the hyperbola $4x^2-3y^2=1$. [*P.U. 1956S*]

9. Tangents are drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from the point (h, k), and make angles θ and θ' with the axis of x ; prove that

$$\tan\theta+\tan\theta'=\frac{2hk}{h^2-a^2}.$$

10. Find the equation of the common tangent to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$. [*P. U.*]

[**Hint.** Let m be the slope of a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Then its equation is $y=mx\pm\sqrt{a^2m^2-b^2}$ ...(1)

If it touches the hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, *i.e.*, $\frac{x^2}{(-b^2)}-\frac{y^2}{(-a^2)}=1$, $\left(\text{Form } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1\right)$,

then $\pm\sqrt{a^2m^2-b^2}=\pm\sqrt{-b^2m^2-(-a^2)}$ ...(2) $[c=\pm\sqrt{a^2m^2-b^2}]$

Square (2), find the values of m, and substitute them one by one in (1).]

11. Find the equations of the straight lines which are tangents both to the parabola $y^2=8x$ and the hyperbola $3x^2-y^2=3$. [*M.U. 1943S*]

12. If the polar of a point (α, β) with respect to the parabola $y^2=4ax$ touches the circle $x^2+y^2=4a^2$, prove that the point (α, β) is on the rectangular hyperbola $x^2-y^2=4a^2$. [*P.U. 1952*]

13. If the polars of (x_1, y_1), (x_2, y_2) with respect to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are at right angles, then will $\frac{x_1x_2}{y_1y_2}+\frac{a^4}{b^4}=0$. [*P.U. 1955S*]

14. The polar of any point on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will touch $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

[**Hint.** Let (x_1, y_1) be any point. The equation of its polar w.r.t. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ ...(1). Find the condition that it may touch $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (Reduce (1) to the form $lx+my+n=0$, and use $a^2l^2+b^2m^2=n^2$ (Art. 126, Ex. 6).) It will be found that $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$, $\therefore$ (x_1, y_1) lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.]

15. If the polar of a point with respect to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then show that the locus of the point is the hyperbola. [*P.U.*]

16. The poles with respect to the parabola $y^2=4ax$ of tangents to the circle $x^2+y^2=a^2$ are on the hyperbola $4x^2-y^2=4a^2$.

[**Hint.** Let (x_1, y_1) be the pole. Then the equation of the polar w.r.t. the parabola is $yy_1=2a(x+x_1)$. If it touches the circle $x^2+y^2=a^2$, the length of the $\perp$ from the centre (0, 0) on it = the radius,

$\therefore \quad \pm\frac{2ax_1}{\sqrt{4a^2+y_1^2}}=a.$ Cancel a. Square. $\therefore$ (x_1, y_1) lies on ?]

17. The poles with respect to the parabola $y^2=4ax$ of tangents to the hyperbola $4x^2-y^2=4a^2$ are on the circle $x^2+y^2=a^2$.

****18.** Prove that the locus of the pole with respect to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ of any tangent to the circle, whose diameter is the line joining the foci, is the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$. [*B. U.*]

****19. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is the hyperbola $a^2x^2-b^2y^2=(a^2+b^2)^2$.**

[*B.U. 1947*]

[**Hint.** The equation of the normal to the hyperbola at $(a\sec\theta, b\tan\theta)$ is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$ ...(1). It meets the x-axis where, putting $y=0$ in (1), $x=\frac{a^2+b^2}{a}\sec\theta$. It meets the y-axis where, putting $x=0$ in (1), $y=\frac{a^2+b^2}{b}\tan\theta$. $\therefore$ the co-ordinates of P are

$$x=\frac{a^2+b^2}{a}\sec\theta \ldots(2),\quad y=\frac{a^2+b^2}{b}\tan\theta \ldots(3)$$

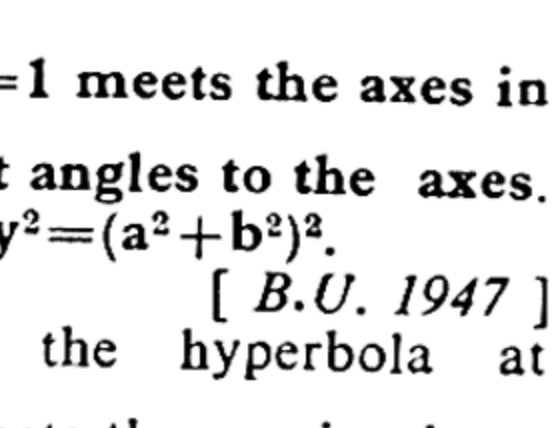
Eliminate θ from (2) and (3) by using $\sec^2\theta-\tan^2\theta=1$.]

20. Shew that the locus of the centre of a circle which touches externally two given circles is a hyperbola. [*Annamalai U.*]

[**Sol.** Let S, S′ be the centres, and a, b the radii of the two given circles, P the centre of the touching circle and r its radius.

Then $S'P-SP=b+r-(a+r)=b-a$ (constant).

$\therefore$ the locus of P is a hyperbola whose foci are S, S′ (Art. 145).]



****21. In the hyperbola $16x^2-9y^2=144$, find the equation to the diameter which is conjugate to the diameter whose equation is $x=2y$.** [*C. U. 1946*]

APPENDIX

EQUATION OF THE TANGENT BY THE METHOD OF DIFFERENTIAL CALCULUS

[To be studied when the student is acquainted with the *geometrical application of the differential coefficient in Differential Calculus.*]

Example. Find the equation of the tangent at the point (x_1, y_1) of the circle $x^2+y^2=a^2$. [Art. 62]

[Method of Differential Calculus.]

The equation of the circle is $x^2+y^2=a^2$.

(*i*) Differentiating both sides w.r.t. x,

$$2x+2y\frac{dy}{dx}=0, \therefore \frac{dy}{dx}=-\frac{x}{y}.$$

(*ii*) Putting $x=x_1$, $y=y_1$, $\frac{dy_1}{dx_1}=-\frac{x_1}{y_1}$.

(*iii*) $\therefore$ the equation of the tangent at (x_1, y_1) is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1) \qquad \left[y-y_1=\frac{dy_1}{dx_1}(x-x_1)\right]$$

or $yy_1-y_1^2=-xx_1+x_1^2$ or $xx_1+yy_1=x_1^2+y_1^2=a^2$

[$\because$ (x_1, y_1) lies on the circle]

$\therefore$ $\mathbf{xx_1+yy_1=a^2}$, which is the required equation.

[**Comparison of the two methods.** The foregoing method (which is accepted in the examination) is quicker than the method of chord (Art. 62).]

[Rule to find the equation of the tangent at the point (x_1, y_1) of a curve whose equation is given :

(*i*) *Differentiate both sides of the equation of the curve w.r.t. x, and find* $\frac{dy}{dx}$.

(*ii*) *Put $x=x_1$, $y=y_1$, and find the value of* $\frac{dy}{dx}$, *i.e.,* $\frac{dy_1}{dx_1}$.

(*iii*) $\therefore$ *the equation of the tangent at (x_1, y_1) is*

$$\mathbf{y-y_1=\frac{dy_1}{dx_1}(x-x_1).}$$

Simplify this equation by means of the condition that (x_1, y_1) lies on the curve. The result is the required equation.]

EXAMPLES

1. Find the equation of the tangent at the point (x_1, y_1) of the circle $x^2+y^3+2gx+2fy+c=0$. [Art. 64]

[*Ans.* $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.]

[**Hint.** After step (*iii*) simplify the equation as in Art. 64, step (*iv*).]

2. Find the equation of the tangent at the point (x_1, y_1) of the parabola $y^2=4ax$. [Art. 99]

[*Ans.* $yy_1=2a(x+x_1)$.]

[**Hint.** After step (*iii*) simplify the equation as in Art. 99, step (*iv*).]

3. Find the equation of the tangent at the point (x_1, y_1) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [Art. 122]

$\left[Ans.\ \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.\right]$

[**Hint.** After step (*iii*) simplify the equation as in Art. 122, step (*iv*).]

4. Find the equation of the tangent at the point (x_1, y_1) of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. [Art. 142, Ex.]

$\left[Ans.\ \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.\right]$

PUNJAB UNIVERSITY PAPERS

1953

1. Find the perpendicular distance of the point (f, g) from the straight line $ax+by+c=0$. Explain your answer.

Find the co-ordinates of the foot of the perpendicular from the point (2, 3) to the straight line $4x-5y+8=0$.

2. Find the equation of the chord of the circle $x^2+y^2=a^2$ whose middle point is (h, k).

Find the angle at which the circles $x^2+y^2-2x-4y=4$ and $x^2+y^2=16$ intersect.

3. Find the equation of the polar of the point (x_1, y_1) to the parabola $y^2=4ax$.

Prove that the polars of all points on the directrix pass through the focus.

4. Find the equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta,\ b\sin\theta)$.

Prove that $lx+my+n=0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}.$$

5. Define hyperbola. Obtain its equation in the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Find the eccentricity, co-ordinates of the foci and the length of the latus rectum of the hyperbola $9x^2-16y^2=144$.

1954

1. Find the area of a triangle the co-ordinates of whose vertices are given.

Use it to find the equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2).

Determine the area of the triangle formed by the lines $x+4y=9$, $9x+10y+23=0$ and $2y+7x=11$.

2. Determine the co-ordinates of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the form $x=a\cos\theta$, $y=b\sin\theta$.

Determine the equation of the tangent to the ellipse at this point.

If this tangent meets the axes in the points P and Q, show that the locus of the mid-point of PQ is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$.

3. Show that the equation $ax^2+ay^2+2gx+2fy+c=0$ always represents a circle. Find its centre and radius.

Find the radical centre of the three circles

$x^2+y^2+4x+6y-3=0$, $3x^2+3y^2+6y+8x-9=0$ and $x^2+y^2-5x=0$.

Find the length of the tangent from this point to any of the circles.

4. Find the co-ordinates of the point of intersection of the tangents drawn to the parabola $y^2=4x$ at the points where it meets the straight line $7y-4x+10=0$.

Show that the locus of the mid-points of a system of parallel chords of a parabola is a straight line parallel to its axis. If this straight line meets the parabola in the point V, show that the tangent at V is parallel to the given system.

5. Tangents are drawn from the point $(-2, -1)$ to the hyperbola $2x^2-3y^2=6$. Find their equations.

Find the equation of the hyperbola whose foci are (2,0) and (−2,0) and eccentricity, $\frac{3}{2}$.

1955

1. (*a*) Find the perpendicular distance of the point (x', y') from the straight line $x \cos \alpha + y \sin \alpha = p$.

(*b*) Find the co-ordinates of the orthocentre of the triangle, the equations of whose sides are $y+x=6$, $3y-x+2=0$ and $5x-3y+2=0$. Also determine the length of the altitude on the side whose equation is $x+y=6$.

2. (*a*) A circle passes through the points (5, 7), (6, 6) and (2, −2). Find the co-ordinates of its centre and length of its radius.

(*b*) Find the equation of the circle described on the common chord of the two circles $x^2+y^2-6x-4y=12$ and $x^2+y^2-2x-6y=15$ as diameter.

3. (*a*) Determine the equation of the chord of contact of tangents drawn from the point (x_1, y_1) to the parabola $y^2=4ax$.

If the point lies on the directrix of the parabola, show that its chord of contact passes through the focus.

(*b*) Show that the line $3y=6x+2$ touches the parabola $3y^2=16x$ and find the co-ordinates of its point of contact.

4. (*a*) Find the equation of the normal at a point $P(x', y')$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If this normal meets the major axis in the point G and PN be perpendicular to the axis, show that $CG=e^2$. CN where C is the centre of the ellipse.

(*b*) Determine the co-ordinates of points on the ellipse $x^2+3y^2=37$ at which the normal is parallel to the line $6x-5y=2$.

5. (*a*) Obtain the equation of the hyperbola in the standard form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Show that the difference of the focal distances of any point on this hyperbola is constant.

(*b*) Find the eccentricity of the hyperbola whose equation is
$2x^2-3y^2=15$.

1956

1. (*a*) Prove that any equation of the first degree in x and y represents a straight line.

Show that $(ax+by+c)+k(a'x+b'y+c')=0$ represents a straight line through the point of intersection of the straight lines $ax+by+c=0$ and $a'x+b'y+c'=0$.

(*b*) Determine the equation of the bisector of the acute angle between the two straight lines whose equations are $4x-3y+2=0$ and $12x+5y-8=0$.

2. Find the co-ordinates of the radical centre of the circles whose equations are $x^2+y^2-3x-5y+8=0$, $x^2+y^2-6x-8y+11=0$ and $2x^2+2y^2-5x-6y+6=0$.

Obtain the length of the tangent drawn from this point to the third circle. Write down the equation of the circle whose centre is the radical centre and radius is the length of the tangent obtained. Show that this circle is orthogonal to any one of the given circles.

3. (*a*) Show that $(at^2, 2at)$ are the co-ordinates of any point on the parabola $y^2=4ax$. Find the equation of the normal to the parabola at this point. Hence, or otherwise, show that three normals can always be drawn from any point to the parabola.

(*b*) Find the length of the chord $4y=3x+8$ intercepted by the parabola $y^2=8x$.

4. (*a*) Prove that the locus of the middle points of a system of parallel chords of an ellipse is a straight line through its centre.

(*b*) Find the equations of the tangents to the ellipse $x^2+3y^2=3$,

(*i*) which are parallel to the straight line $y=4x+8$.

(*ii*) which are perpendicular to the straight line $x=4y+5$.

5. (*a*) Show that the straight line given by the equation $y=mx+\sqrt{a^2m^2-b^2}$ always touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Determine the locus of the point of intersection of two tangents to the above hyperbola which are at right angles to each other.

(*b*) Find the equation to the hyperbola whose eccentricity is $\frac{5}{4}$, focus is the point $(a, 0)$ and directrix is the line $4x+3y=a$.

1957

1. (*a*) Find the equation of a straight line in the form

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}=r.$$

State clearly what is represented by (x', y'), θ and r.

(*b*) Find the equation of the straight line through $(2, -3)$ perpendicular to $7x-5y+23=0$.

(*c*) Find the equations of the straight lines through $(1, -1)$ inclined at an angle of 60° to $\sqrt{3}\,x-y+7=0$. 4, 3, 5

2. (*a*) Find the condition that $lx+my+n=0$ may touch the circle $x^2+y^2=a^2$.

(*b*) Find the equation of the circle, which has for a diameter the common chord of the circles

$x^2+y^2+4x+5y+1=0$ and $x^2+y^2-6x+3y+2=0$. 5, 7

3. (*a*) (*i*) Find the equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which has (x', y') as its middle point.

(*ii*) If chords of an ellipse pass through a fixed point, their mid-points lie on another ellipse.

(*b*) Find the eccentricity, the co-ordinates of the foci and the length of the latus rectum of the ellipse $4x^2+9y^2=1$. 5, 4, 3

4. (*a*) Find the condition that (t_1), (t_2) be the ends of a focal chord of parabola $y^2=4ax$. Show that the tangents at the ends of a focal chord of a parabola intersect at right angles on its directrix.

(*b*) Find the equation of the locus of the point of intersection of two tangents to a parabola, which make an angle of 45° with one another. 8, 7

5. (*a*) Show that the locus of the pole of any tangent of the circle $x^2+y^2=a^2$ with respect to the hyperbola $x^2-y^2=a^2$ is the circle itself.

(*b*) Show that the area of the triangle formed by the lines $x-y=0$, $x+y=0$ and any tangent to the hyperbola $x^2-y^2=a^2$ is constant. 7, 8

Answers and Hints

CHAPTER I

Page 3. Ex. (*i*) A(0, 0) ; B(2*a*, 0) ; C(2*a*, 2*a*) ; D(0, 2*a*).
(*ii*) A($-a$, $-a$) ; B(a, $-a$) ; C(a, a) ; D($-a$, a).

Pages 5-6.

1. (*a*) $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, where (x_1, y_1), (x_2, y_2) are the two pts.
(*b*) (*i*) 13 ; (*ii*) $(a+b)\sqrt{2}$.

2. (*i*) $a(t_2-t_1)\sqrt{(t_2+t_1)^2+4}$; (*ii*) $2a \sin \frac{\alpha-\beta}{2}$.

[**Hint.** (*ii*) $a \cos \beta - a \cos \alpha = a(\cos \beta - \cos \alpha) = 2a \sin \frac{\beta+\alpha}{2} \sin \frac{\alpha-\beta}{2}$,

$a \sin \beta - a \sin \alpha = a(\sin \beta - \sin \alpha) = 2a \cos \frac{\beta+\alpha}{2} \sin \frac{\beta-\alpha}{2}$

$= -2a \cos \frac{\beta+\alpha}{2} \sin \frac{\alpha-\beta}{2}$.]

[****Note.** It is assumed in (*i*) that $t_2 > t_1$ so that the given answer gives a positive result for the distance. Similarly it is assumed in (*ii*) that $\alpha > \beta$.]

3. 7 or -1. **4.** [**Hint.** Let the ordinate $= y$.]

7. [**Hint.** Let A(0, -1), B(2, 1), C(0, 3), D(-2, 1) be the pts. Show that AB = BC = CD = DA,
and $AC^2 = AB^2 + BC^2$ so that $\angle ABC$ = 1 rt. $\angle$.]

8. [**Hint.** Let A(4, 3), B(-2, 3), C(6, -1) be the vertices of the triangle and S the pt. (1, -1). Show that SA = SB = SC.]

Pages 9-10.

1. (*a*) (*i*) $x = \frac{m_1x_2+m_2x_1}{m_1+m_2}$, $y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$, where (x_1, y_1), (x_2, y_2) are the two pts., and $m_1 : m_2$ the ratio in which their join is divided internally.

(*ii*) $x = \frac{m_1x_2-m_2x_1}{m_1-m_2}$, $y = \frac{m_1y_2-m_2y_1}{m_1-m_2}$, where (x_1, y_1), (x_2, y_2) are the two pts., and $m_1 : m_2$ the ratio in which their join is divided externally.

(*b*) (0, 0) ; (3, -9).

2. (*a*) $x = \frac{la_2+ma_1}{l+m}$, $y = \frac{lb_2+mb_1}{l+m}$. [**Hint.** Proceed as in Art. 6-I.]

4. [**Hint.** Let A(1, -1), B(-2, 2), C(4, 8), D(7, 5) be the pts. **Test for a rectangle.** ABCD is a ||gm., and $\angle ABC$ = 1 rt. $\angle$.]

Page 12. 2. (0, 3).

3. [**Hint.** Let A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) be the vertices of the

triangle. Proceed as in solved Ex. 1, Art. 7, and prove that the co-ordinates of G, the pt. which divides the median AD in the ratio 2 : 1, are

$$x=\frac{x_1+x_2+x_3}{3},\quad y=\frac{y_1+y_2+y_3}{3}.$$

The *symmetry** of these results shows that the co-ordinates of the pts. which divide the other medians in the ratio 2 : 1 are the same as those of G. ∴ G lies on each median, *i.e.*, the medians are concurrent.]

5. $(-1, 0)$.

6. [**Hint.** Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle. Proceed as in solved Ex. 2, Art. 7, and prove that the co-ordinates of I, the pt. of intersection of AD and BE, the bisectors of the ∠s A and B, are $x=\frac{ax_1+bx_2+cx_3}{a+b+c},\ y=\frac{ay_1+by_2+cy_3}{a+b+c}$.

The *symmetry*† of these results shows that the co-ordinates of the pt. of intersection of the bisectors of the ∠s B and C are the same as those of I. ∴ I lies on each bisector, *i.e.*, the bisectors of the ∠s are concurrent.]

Page 17.

1. (*a*) $\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]$, where (x_1, y_1), (x_2, y_2), (x_3, y_3) are the vertices of the triangle.

(*b*) (*i*) 11 ; (*ii*) Area in magnitude $=a^2$.

[****Note.** The negative sign in the Ans. $(-a^2)$ means that, plotting the vertices of the triangle and going round the triangle in the given order of the vertices, the area is on the right hand.]

2. (*a*) $\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)$.]

(*b*) (*i*) $2ab\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}$.

[**Hint.** By the Rule (Art. 9), it will be found that

$$\triangle=\tfrac{1}{2}ab[\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\alpha-\gamma)]$$

$$=\tfrac{1}{2}ab\left[2\sin\frac{\gamma-\alpha}{2}\cos\frac{2\beta-\alpha-\gamma}{2}-2\sin\frac{\gamma-\alpha}{2}\cos\frac{\gamma-\alpha}{2}\right]$$

$$=ab\sin\frac{\gamma-\alpha}{2}\left[\cos\frac{2\beta-\alpha-\gamma}{2}-\cos\frac{\gamma-\alpha}{2}\right]$$

---

*"*The symmetry of these results shows*" means that "on changing x_1, x_2, x_3 ; y_1, y_2, y_3 *cyclically, i.e.,* on changing x_1 to x_2, x_2 to x_3, x_3 to x_1 ; y_1 to y_2, y_2 to y_3, y_3 to y_1, the results remain unchanged".

For, $x=\frac{x_1+x_2+x_3}{3}$ becomes $x=\frac{x_2+x_3+x_1}{3}=\frac{x_1+x_2+x_3}{3}$.

Similarly for $y=\frac{y_1+y_2+y_3}{3}$.

†"*The symmetry of these results shows*" means that "on changing x_1, x_2, x_3 ; y_1, y_2, y_3 ; a, b, c *cyclically, i.e.,* on changing x_1 to x_2, x_2 to x_3, x_3 to x_1 ; y_1 to y_2, y_2 to y_3, y_3 to y_1 ; a to b, b to c, c to a, the results remain unchanged."

For, $x=\frac{ax_1+bx_2+cx_3}{a+b+c}$ becomes $x=\frac{bx_2+cx_3+ax_1}{b+c+a}=\frac{ax_1+bx_2+cx_3}{a+b+c}$.

Similarly for $y=\frac{ay_1+by_2+cy_3}{a+b+c}$.

$$=ab \sin\frac{\gamma-\alpha}{2}.\ 2\sin\frac{\beta-\alpha}{2}\sin\frac{\gamma-\beta}{2}.\ ]$$

(*ii*) Area in magnitude$=a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)$.
[**Note. The negative sign in the Ans. $[-a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)]$ means that, plotting the vertices of the triangle and going round the triangle in the given order of the vertices, the area is on the right hand.]
[**Hint.** $\triangle=\frac{1}{2}.2a^2(t_1^2t_2-t_2^2t_1+t_2^2t_3-t_3^2t_2+t_3^2t_1-t_1^2t_3]$
$=a^2[t_1t_2(t_1-t_2)+t_2t_3(t_2-t_3)+t_3t_1(t_3-t_1)]$
$=-a^2(t_1-t_2)(t_2-t_3)(t_3-t_1).]$
[(From Algebra) $\because\ \Sigma ab(a-b)=-(a-b)(b-c)(c-a)$]

3. $\frac{a^2}{2}(t_1-t_2)(t_2-t_3)(t_3-t_1)$.

Page 18. Art. 10. 3. [**Hint.** The three pts. (x, y), $(a, 0)$, $(0, b)$ are collinear.]

MISCELLANEOUS EXAMPLES ON CHAPTER I

Pages 18-20. 2. $\left(\frac{1+\sqrt{3}}{2}, \frac{7-5\sqrt{3}}{2}\right)$, or $\left(\frac{1-\sqrt{3}}{2}, \frac{7+5\sqrt{3}}{2}\right)$.

3. $(0, 2\sqrt{3})$, or $(3, -\sqrt{3})$. **4.** $(3, -4)$. **5.** $(5, -4)$; $(7, 0)$.

6. $\left(-\frac{1}{2}, \frac{5}{2}\right)$; $(1, 1)$; $\left(\frac{5}{2}, -\frac{1}{2}\right)$. **8.** 5.

9. $\left(\frac{10}{3}, \frac{7}{3}\right)$; $\left(\frac{13}{4}, \frac{9}{4}\right)$.

[**Hint.** (*ii*) Let A(2, 1), B(5, 2), C(3, 4) be the vertices of the triangle and (x, y) the required co-ordinates of the circumcentre S.
Then SA=SB=SC, ∴ what ? Square, and solve as in Ex. 2.]

10. [**Hint.** Choose the axes as in Art. 8-II. Then C is $(a, 0)$, B is $(-a, 0)$, and let A be (x_1, y_1).
Or thus : Let A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) be the vertices of the triangle.]

11. 5 : 3.

12. $(-2, -8)$.
[**Hint.** Let the required co-ordinates of the other end be (x, y). The mid-pt. of the join of $(-6, 4)$ and (x, y) is ?
This is the same as $(-4, -2)$.]

13. (11, 10).

14. (*b*) [**Hint.** Let ABC be the triangle. Choose the axes as in Art. 8-II. Then C is $(a, 0)$, B is $(-a, 0)$, and let A be (x_1, y_1).]

15. $\frac{a^2}{2t_1t_2t_3}(t_1-t_2)(t_2-t_3)(t_3-t_1)$.

16. (7, 2), or (1, 0). [**Hint.** Use the complete area formula.] **18.** 11.

19. Area in magnitude$=\frac{41}{2}$.

[**Note. The negative sign in the Ans. $\left(-\frac{41}{2}\right)$means that, plotting the vertices of the quadrilateral and going round the quadrilateral in the given order of the vertices, the area is on the right hand.]

CHAPTER II

Page 21. **2.** (3, −4), (−4, −3).

Pages 22-23. **1.** $x^2+y^2-4y=0$.

2. $2x+y+5=0$.

[**Hint.** (*i*) Let (x, y) be *any* pt. on the locus.

(*ii*) Then distance of (x, y) from $(1, -2)$ = distance of (x, y) from $(-3, -4)$.

(*iii*) $\therefore \sqrt{(x-1)^2+(y+2)^2} = \sqrt{(x+3)^2+(y+4)^2}$.

Square and simplify.]

3. $3x^2+3y^2+2x+16y+15=0$.

4. $x^2+y^2-a^2=0$. [$1-k^2$ cancels thro' out.]

6. $2ax+k^2=0$. **7.** $mx-y=0$. **8.** $x^2-3y^2=0$.

9. $y^2-2ax+a^2=0$. **10.** $2x+3y-1=0$, or $2x+3y-9=0$.

CHAPTER III

Page 26. **2.** (*i*) $x-3=0$; (*ii*) $x+4=0$. **4.** $x+2=0$.

Page 27. **1.** (*i*) $\sqrt{3}x-y=0$; (*ii*) $x=0$. [**Hint.** The line is the y-axis.] (*iii*) $\sqrt{3}x+y=0$; (*iv*) $x+\sqrt{3}y=0$. **2.** $x-y=0$, $x+y=0$.

Pages 28-29. **1.** (*b*) (*i*) $x-y+2=0$; (*ii*) $x-y-3=0$.

2. $3x-4y-8=0$. **4.** $x-y-3=0$, or $x+y+3=0$.

Pages 30-31. **1.** (*a*) $\frac{x}{a}+\frac{y}{b}=1$, where a, b are the intercepts of the line on the axes. (*b*) $3x+4y-12=0$. **2.** (*i*) $y=mx+c$.

3. $x+y-7=0$. **4.** $x-y+1=0$. [**Hint.** Let the intercepts be $a, -a$.]

5. $x+y-1=0$, $x-y-3=0$. [**Hint.** The intercepts are (*i*) equal, (*ii*) equal in magnitude but opposite in sign.]

8. $x-2y+6=0$.

[**Hint.** (*i*) Let the required equation of the line be $\frac{x}{a}+\frac{y}{b}=1$...(1)

(*ii*) The pt. which divides the join of $(a, 0)$ and $(0, b)$ in the ratio 1 : 2 is $\left(\frac{2a}{3}, \frac{b}{3}\right)$ ...(2). This is the same as $(-4, 1)$ ...(3)

Compare (2) and (3).]

Page 32. **3.** (*i*) $\sqrt{3}x-y+4=0$; (*ii*) $x-\sqrt{3}y-4=0$.

4. (*i*) m is the tangent of the angle which the part of the line above the x-axis makes with the x-axis and c the intercept of the line on the y-axis ; (*ii*) a and b are the intercepts of the line on the axes ; (*iii*) p is the length of the perpendicular from the origin upon the line and α the angle which this perpendicular makes with the x-axis.

Pages 34-35. Art. 24. **1.** [**Hint.** Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be *any* three pts. on the locus of the equation $lx+my-1=0$ ...(1). Then the co-ordinates of these pts. satisfy (1). Get three equations in l, m ; (2), (3), (4). Solve (2) and (3) for l, m by cross-multiplication, and substitute these values of l, m in (4), etc.]

3. [**Hint.** Choose the axes as in Art. 8-I, Rule I. It will be found that the equation of the locus is $x+y=a$, which, being an equation of the first degree in x and y, represents a st. line (Art. 24).]

4. [**Hint.** Choose the axes as in Art. 8-II, Rule II.]

5. $-3 : 2$.

[**Note.** The negative sign means that the join of the two pts. is divided by the given line externally in the ratio 3 : 2. (*Cf.* Art. 6-II, Note.)]

6. $-2 : 5$; $1 : 2$.

[**Note.** The negative sign means that the join of the two pts. is divided by the x-axis externally in the ratio 2 : 5.]

Page 35. Art. 25. 1. (*i*) $y=\frac{2}{3}x+2$. (*ii*) $y=-\frac{1}{2}x+\frac{3}{2}$.

3. (*a*) $-\cot\alpha$. (*b*) (*i*) 135° ; (*ii*) 150° ; (*iii*) $\tan^{-1}\frac{1}{2}$.

4. (*i*) -1 ; (*ii*) 1.

Page 36. 1. $\frac{x}{-\frac{5}{3}}+\frac{y}{\frac{5}{4}}=1$.

2. (*i*) $-5, 4$; (*ii*) $2\sin\alpha, 2\cos\alpha$.

[**Hint.** Use the Rule (Art. 26, Cor.).]

Pages 38-39. Art. 27.

1. (*i*) $x\cos 240°+y\sin 240°=2$. (*ii*) $\frac{5}{13}x-\frac{12}{13}y=1$.

(*iii*) $x\cos 180°+y\sin 180°=1$. (*iv*) $x\cos 90°+y\sin 90°=\frac{3}{2}$.

2. $x\cos 225°+y\sin 225°=\frac{5}{\sqrt{2}}$; $p=\frac{5}{\sqrt{2}}$, $\alpha=225°$. **3.** $\theta=60°$, $p=2$.

4. $-\frac{m}{\sqrt{1+m^2}}x+\frac{1}{\sqrt{1+m^2}}y-\frac{c}{\sqrt{1+m^2}}=0$, $p=\frac{c}{\sqrt{1+m^2}}$;

$\frac{b}{\sqrt{a^2+b^2}}x+\frac{a}{\sqrt{a^2+b^2}}y-\frac{ab}{\sqrt{a^2+b^2}}=0$, $p=\frac{ab}{\sqrt{a^2+b^2}}$.

Page 39. Art. 28. 2. $3x-y-1=0$. **3.** $x\cos\alpha+y\sin\alpha=p$.

Page 41.

1. (*a*) $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$, where (x_1, y_1), (x_2, y_2) are the two pts.

(*b*) $x+3y+7=0$, $3x-y+1=0$, $7x+y-11=0$.

[**Hint.** Let A(1, 4), B(2, −3), C(−1, −2) be the vertices of the triangle. Find the equations of the sides BC, CA, AB.]

2. (*i*) $xy_1-x_1y=0$. (*ii*) $x+y-(a+b)=0$. (*iii*) $x+t_1t_2y-a(t_1+t_2)=0$.

3. (*i*) $y-4=0$.

[**Hint.** (*ii*) It will be found that the equation of the median is $y=\frac{2}{0}(x-1)$. Cross-multiply, then $y(0)=2(x-1)$ or $x-1=0$.

Note. Important. *If in the two-pt. form, you get a zero in the denominator, cross-multiply.*]

4. $\frac{x}{a}+\frac{y}{b}=1$. **5.** $3, -\frac{3}{2}$. **6.** $-\sqrt{3}$, 120°.

7. 1 : 1, *i.e.*, the line is bisected.

9. $3x-4y+3=0$; $\frac{5}{4}$.

[**Hint.** (*ii*) Let the line meet the axes in A, B. Find the intercepts OA, OB on the axes. Then $AB=\sqrt{OA^2+OB^2}$.]

Pages 43-44. 1. (x_1, y_1) is the given pt. thro' which the line is drawn, θ the angle which the line makes with the x-axis, and r the distance of the pt. (x, y) from (x_1, y_1). 2. 2.

3. $\frac{11}{2\sqrt{2}}$.

4. (*a*) $x+y-7=0$; (*b*) (2, 5), (4, 3).

[**Hint.** (*b*) (*i*) $r=\sqrt{2}$; (*ii*) $r=-\sqrt{2}$.]

5. 15° or 75°.

Page 45. **Ex.** $\pm\frac{C}{\sqrt{A^2+B^2}}$.

Page 49. 1. (*i*) 60° ; (*ii*) 45° ; (*iii*) 90° ; (*iv*) $\tan^{-1}\frac{7}{4}$.

2. $\frac{ab'-a'b}{aa'+bb'}$. 3. $\tan^{-1}3$. 4. 45°. 5. $\tan^{-1}\frac{7}{6}$ 6. 45°.

Page 51. 2. [**Hint.** Choose the axes as in Art. 8-II.
Or thus : Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle. Use the slope formula for the line passing thro' two pts. (Art. 29, Cor.).]

Pages 52-53. 3. $3x-4y+6=0$. 4. $2x-3y+7=0$.

5. $4x+y-3=0$. 6. $x-3y+4=0$.

Page 54. 1. (*a*) $\tan^{-1}\frac{a'b-ab'}{aa'+bb'}$.

3. [**Hint.** Here m_1=slope of the line passing thro' (1, 2) and (2, −2)

$$=\frac{-2-2}{2-1} \qquad \left[\frac{y_2-y_1}{x_2-x_1}\right]$$

$$=-4,$$

m_2=slope of the line passing thro' (8, 2) and (4,1)

$$=\frac{1-2}{4-8}=\frac{-1}{-4}=\tfrac{1}{4}.$$]

Pages 55-56. 3. (*i*) $3x+2y-17=0$.

[**Hint.** First write the equation of the given line in the general form : $2x-3y+1=0$. (Form $Ax+By+C=0$)]

(*ii*) $xx'-yy'-(x'^2-y'^2)=0$.

4. (*i*) $mx-ly-(mh-lk)=0$. (*ii*) $qx+py-(qa+pb)=0$.

5. $12x+5y+3=0$.

[**Hint.** Use Method II of solved Ex., Art. 37.

The slope of $CA=\frac{4-(-1)}{10-(-2)}$ $\qquad \left[\frac{y_2-y_1}{x_2-x_1}\right]$

$$=\frac{5}{12}.$$]

6. $2x+4y-11=0$.

Page 57. 1. $3x+y-17=0$, $x-3y+11=0$.

2. $y+2=0$, $\sqrt{3}x-y-(2+3\sqrt{3})=0$.

Pages 58-59. **2.** (*i*) $(2, -2)$. (*ii*) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

3. $\left(-\frac{1}{10}, \frac{37}{10}\right)$.

[**Hint.** The foot of the $\perp$ is the pt. of intersection of the line $y=3x+4$ and the line thro' $(2, 3)$ $\perp$ to $y=3x+4$.]

4. $x+y-3=0$.

5. $\left(\frac{11}{13}, -\frac{7}{13}\right)$.

[**Hint.** Let A(1, 0), B(2, −4), C(−5, −2) be the vertices of the triangle. Find the equation of the line thro' A $\perp$ to BC, and then the equation of the line thro' B $\perp$ to CA. Find the pt. of intersection of these two lines.]

6. (*i*) c^2 ; (*ii*) $\frac{361}{48}$.

8. $\left(\frac{p}{1+\sin\alpha+\cos\alpha}, \frac{p}{1+\sin\alpha+\cos\alpha}\right)$.

[**Hint.** Draw a rough Fig. The equations of the sides are

$y=0$ ... (1), $x\cos\alpha+y\sin\alpha-p=0$ ...(2), $x=0$ ... (3)

Solving (2) and (3), then (3) and (1), and then (1) and (2), the vertices are A $\left(0, \frac{p}{\sin\alpha}\right)$, B(0,0), C$\left(\frac{p}{\cos\alpha}, 0\right)$.

$\therefore$ the lengths of the sides are

$$a=\mathrm{BC}=\frac{p}{\cos\alpha},$$

$$b=\mathrm{CA}=\sqrt{\frac{p^2}{\cos^2\alpha}+\frac{p^2}{\sin^2\alpha}}=\sqrt{\frac{p^2(\sin^2\alpha+\cos^2\alpha)}{\cos^2\alpha\sin^2\alpha}}=\frac{p}{\cos\alpha\sin\alpha},$$

$c=\mathrm{AB}=\frac{p}{\sin\alpha}$. Now use the incentre formulae :

$$x=\frac{ax_1+bx_2+cx_3}{a+b+c},\quad y=\frac{ay_1+by_2+cy_3}{a+b+c}.$$ (Art. 7, solved Ex. 2)]

9. $(-1, 8)$.

Page 60. **2.** 4. **3.** −2. [**Hint.** Solve the first and third equations (free from the unknown quantity a), and substitute these values of x and y in the second equation.]

4. $m_1(c_2-c_3)+m_2(c_3-c_1)+m_3(c_1-c_2)=0$.

Page 66. **1.** $64x-23y-59=0$.

2. (*i*) $\frac{ax+by+c}{ax'+by'+c}=\frac{a'x+b'y+c'}{a'x'+b'y'+c'}$. (*ii*) $(ac'-a'c)x+(bc'-b'c)y=0$.

3. $6x+13y-25=0$. [Divide by 43 the equation obtained by Method I.]

4. $5x+3y+8=0$. [Divide by 28 the equation obtained by Method I.]

Page 67. **Ex.** $h\cos\alpha+k\sin\alpha-p$. [**Hint.** Proceed as in Art. 44-I.]

Pages 69-70. **1.** (*a*) $\frac{a\alpha+b\beta+c}{\sqrt{a^2+b^2}}$. [**Hint.** Proceed as in Art. 44-II.]

(*b*) (*i*) 1. (*ii*) $\frac{a^2+ab-b^2}{\sqrt{a^2+b^2}}$. 2. (*a*) At a distance $\sqrt{h^2+k^2}$.

4. $\frac{(a-b)^2}{\sqrt{a^2+b^2}}$. 5. $5 ; \left(\frac{11}{17}, -\frac{7}{17}\right)$. 6. $\frac{5}{\sqrt{13}}, \frac{5}{\sqrt{13}}, \frac{5}{\sqrt{2}}$.

8. (*i*) $\frac{24}{5}$. (*ii*) $\frac{7}{2}$. 9. (*i*) $\frac{12}{5}$. (*ii*) $\frac{7}{\sqrt{5}}$.

10. (*i*) $\frac{c-c'}{\sqrt{a^2+b^2}}$. (*ii*) $\frac{c-d}{\sqrt{1+m^2}}$.

Page 71. **2.** On opposite sides.

4. [**Hint.** Find the vertices, and show that each vertex and the origin are on the same side of the opposite side of the triangle.]

Pages 72-75. **Art. 46.** **1.** (*a*) $\frac{ax+by+c}{\sqrt{a^2+b^2}} = \pm \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$.

Writing the equations of the lines so that the constant terms c, c_1 are both +ve, the equation of the bisector of the angle in which the origin lies, is

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}},$$

and that of the bisector of the angle in which the origin does not lie, is

$$\frac{ax+by+c}{\sqrt{a^2+b^2}} = -\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}.$$

3. $8x-14y+37=0$, $112x+64y-67=0$.
4. $3x-11y=0$, $99x+27y-130=0$.
5. $4x-7y+5=0$, $7x+4y+90=0$; $4x-7y+5=0$. 7. $x-y-2=0$.
9. $3x-y-18=0$.

MISCELLANEOUS EXAMPLES ON CHAPTER III

Pages 75-79. **2.** $x+y=4$.

[**Hint.** Let OA$=a$, OB$=b$, then $a+b=8$ ... (1). Take OA, OB as axes. Then A is $(a, 0)$, B is $(0, b)$

$\therefore$ the co-ordinates of the mid-pt. of AB are $x=\frac{a}{2}$, $y=\frac{b}{2}$ ...(2)

Eliminate a, b from (1) and (2) [by substituting the values of a, b from (2) in (1)].]

3. $\frac{h}{x}+\frac{k}{y}=1$.

[**Hint.** Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$.

Then $\because$ it passes thro' (h, k),

$$\therefore \frac{h}{a}+\frac{k}{b}=1 \qquad \ldots(1)$$

The co-ordinates of R are $x=a$, $y=b$ ...(2)]

5. $y=\sqrt{3}x-2$ or $y=x \tan 60^\circ -2$, the line makes an angle of 60° with the x-axis, and makes an intercept (-2) on the y-axis ;

$\frac{x}{3}+\frac{y}{-\frac{3}{2}}=1$, the line makes intercepts 3, $-\frac{3}{2}$ on the axes ; $x\cos 300°+y\sin 300°=2$, the length of the $\perp$ from the origin upon the line=2, and this $\perp$ makes an angle of 300° with the x-axis.

7. $2bx-2a'y+a'b'-ab=0$. **8.** $-\frac{1}{4}, \frac{1}{3}$.

9. $-\frac{Ax_1+By_1+C}{A\cos\theta+B\sin\theta}$. **10.** $\tan^{-1}\frac{1}{2}$.

11. [**Hint.** The equation of the given line is

$$\frac{x}{\cos\theta}+\frac{y}{\sin\theta}=a \text{ or } x\sin\theta+y\cos\theta-a\sin\theta\cos\theta=0\ .]$$

12. $x+y=0$, $x+y+16=0$. **13.** $y-k=\frac{2m}{1-m^2}(x-h)$, $y-k=0$.

14. Sides : $2x+y-4=0$, $x-2y+3=0$, $2x+y-14=0$, $x-2y+13=0$.
Diagonals : $3x-y-1=0$, $x+3y-17=0$.

[**Hint.** Two sides pass thro' (1, 2), and make an angle of 45° with the diagonal passing thro' (1, 2) and (3, 8). The other two sides pass thro' (3, 8), and make an angle of 45° with the above diagonal. The second diagonal passes thro' the mid-pt. of the join of (1, 2) and (3, 8), and is $\perp$ to this join.]

15. (*i*) $\left[\frac{a}{m_1m_2},\ a\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right]$; (*ii*) $\left(a\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}},\ a\frac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}\right)$.

[**Hint.** (*ii*) Solve the two equations by cross-multiplication, and use

$$\sin\theta_2-\sin\theta_1=2\cos\frac{\theta_2+\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2},$$

$$\cos\theta_1-\cos\theta_2=2\sin\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_2-\theta_1}{2},$$

$$\sin\theta_2\cos\theta_1-\cos\theta_2\sin\theta_1=\sin(\theta_2-\theta_1)=2\sin\frac{\theta_2-\theta_1}{2}\cos\frac{\theta_2-\theta_1}{2}.$$

Cancel the common factor $\sin\frac{\theta_2-\theta_1}{2}$ from each denominator.]

16. (1, 1), 45°. (See Art. 33-II, Ex. 5, Note.)

17. (1, 2), $\tan^{-1}\frac{1}{8}$.

18. $x(b-b')-y(a-a')+ab'-a'b=0$, $x(b-b')+y(a-a')+a'b'-ab=0$.
[**Hint.** Draw roughly the lines, and find the co-ordinates of the vertices of the rectangle.]

19. (*a*) $u+kv=0$ is the equation of *any* line thro' the pt. of intersection of the lines $u=0$, $v=0$. (*b*) $23x+23y-11=0$.

20. $x-y=0$; $4x-y=0$.
[**Hint.** Find the pts. of intersection of the line with the axes and the pts. of trisection of their join.]

21. $ax+by-a^2=0$.
22. Area in magnitude $=\frac{1}{2}(a-b)(b-c)(c-a)$.
23. $x^2+y^2-hx-ky=0$. **25.** $x-y=0$.
26. $\left(\frac{5}{2}, \frac{5}{2}\right)$.

27. $17x+1=0$. [**Hint.** Find the pt. of intersection of the two given lines, and then proceed as in Ex. 4, Art. 16.

Or thus : The equation of *any* line thro' the pt. of intersection of the two given lines is $2x-3y+4+k(3x+4y-5)=0$,
i.e., $(2+3k)x+(4k-3)y+4-5k=0$ ...(1)

If it is ‖ to the y-axis, its equation is of the form $x=a$,
∴ coeff. of $y=0$, find the value of k. Substitute this value of k in (1).]

28. $x-1=0$.
[**Hint.** Find the pt. of intersection of the last two lines. Now find the equation of the line thro' the pt. of intersection of the first two lines and passing thro' the pt. of intersection of the last two lines, found above.]

31. 3. **32.** $4x+3y=0$; 1. **33.** $\frac{2+\sqrt{3}}{2}$ **34.** $\frac{36}{\sqrt{10}}, \frac{36}{\sqrt{34}}, \frac{18}{\sqrt{13}}$.

35. $3x+4y-7=0$, $3x+4y+10=0$; $\frac{17}{5}$.

36. $\left(\frac{7}{9}, \frac{14}{27}\right)$; $\frac{14}{27}$.

[**Hint.** Inradius = length of the ⊥ from the incentre on any side (say the first).]

37. $\left(a \pm \frac{a}{b}\sqrt{a^2+b^2}, 0\right)$. [**Hint.** Let the required pt. be $(x, 0)$.]

38. $\left(3, \frac{7}{2}\right), \left(-\frac{1}{3}, -\frac{3}{2}\right)$.

[**Hint.** Let the required pt. be (x, y). Then $3x-2y-2=0$ ...(1)

$\pm\frac{3x+4y-8}{5}=3$ ...(2). Solve (1) with each equation of (2).]

39. $x+7y+5(\sqrt{2}+1)=0$, $x+7y-5(\sqrt{2}-1)=0$.
40. $y-1=0$, $4x-3y+3=0$. **41.** $x-y-12=0$, $7x+7y-36=0$.
42. $3x-y=0$.
44. [**Hint.** Find the equations of the sides of the triangle, and show that each vertex and the origin are on the same side of the opposite side of the triangle.]

CHAPTER IV

Page 81. **2.** (*i*) Two st. lines passing thro' the origin and making angles of 45° and 135° with the x-axis.
**[*Or*, two st. lines bisecting the angles between the axes.]
(*ii*) The y-axis and the st. line passing thro' the origin and making an angle of 45° with the x-axis.

3. (*i*) Two st. lines ‖ to the x-axis and at distances a and $-a$ from it.
(*ii*) The x-axis and the st. line ‖ to the y-axis and at a distance a from it.

4. (*i*) Two st. lines : $x+y+a=0$, and $x+y-a=0$.
(*ii*) Two st. lines : $x+y+a=0$, and $x-y+a=0$.

5. $6x+5y-56=0$, $6x-5y+14=0$.

[**Hint.** Find the separate equations of the sides (by factorising (the L. H. S.'s of) the given equations). Draw a rough Fig., and find the co-ordinates of the vertices. Now find the equations of the diagonals (two-pt. form).]

6. 90°.

[**Hint.** The equation represents the two st. lines $x-a=0$, and $y-b=0$. Use angle formula II (Art. 33-II). See Note in Art. 33-II.

Or thus : The equation represents the two st. lines $x=a$, and $y=b$. Draw a rough Fig. It will be found that one st. line is ‖ to the y-axis and the other ‖ to the x-axis, ∴ the angle between them=90°.]

7. $\sqrt{3}x^2-4xy+\sqrt{3}y^2=0$.

[**Hint.** First find the separate equations of the lines and then their combined equation. Use the Rule (Art. 47, Cor.)]

Pages 83-84. **1.** $4(hb'+h'a)(bh'+a'h)+(aa'-bb')^2=0$.

3. [**Hint.** Let the required equation of a line thro' the origin be

$$y=mx \quad \ldots(1)$$

Then the distance of (x_1, y_1) from (1)=δ [Given]

$$\therefore \pm\frac{y_1-mx_1}{\sqrt{1+m^2}}=\delta \quad \ldots(2)$$ [Complete ⊥ distance formula]

(See Note after Art. 44-II, Cor.)

Eliminate m from (1) and (2) by substituting its value $\left(m=\frac{y}{x} \right)$ from (1) in (2).]

5. [**Hint.** Let the separate equations of the lines $ax^2+2hxy+by^2=0$ be $y=m_1x$ …(1), and $y=m_2x$ …(2)

Then $m_1+m_2=-\frac{2h}{b}$, and $m_1m_2=\frac{a}{b}$ …(3) [Art. 49, Cor. 1]

The equation of the third given line is $lx+my+n=0$ …(4)
Find the area of the triangle whose sides are (1), (2), (4).

It will be found that the area $=\frac{1}{2}n^2\frac{(m_1-m_2)}{(l+mm_1)(l+mm_2)}$

$=\frac{1}{2}n^2\frac{\sqrt{(m_1+m_2)^2-4m_1m_2}}{l^2+lm(m_1+m_2)+lmm_1m_2}$. Substitute from (3).]

Page 86. **1.** (*i*) $x-3y=0$, and $x-4y=0$; $\tan^{-1}\frac{1}{13}$.

(*ii*) $2x+y=0$, and $2x+11y=0$; $\tan^{-1}\frac{4}{3}$.

[**Or thus : Hint.** (*ii*) *Solve the equation as a quadratic in x thus* :

$$x=\frac{-24y\pm\sqrt{576y^2-176y^2}}{8}=\frac{-24y\pm20y}{8}=-\frac{y}{2}, \text{ or } -\frac{11}{2}y.]$$

2. (*i*) $x+y(\sec\theta-\tan\theta)=0$, and $x+y(\sec\theta+\tan\theta)=0$; θ.
(*ii*) $x-y\tan\alpha=0$, and $x+y\cot\alpha=0$; $90°$.
(*iii*) $x\sin\theta-y(\cos\theta+\sqrt{\cos 2\theta})=0$, and $x\sin\theta-y(\cos\theta-\sqrt{\cos 2\theta})=0$

$\tan^{-1}\left[\dfrac{\sqrt{\cos 2\theta}}{\sin\theta}\right]$.

[**Hint.** (*i*) (*b*) Proceed as in solved Ex., Art. 50. (*ii*) It will be found that $x+y(\cot 2\alpha-\operatorname{cosec} 2\alpha)=0$ ...(2),
and $x+y(\cot 2\alpha+\operatorname{cosec} 2\alpha)=0$ ...(3)

Now $\cot 2\alpha-\operatorname{cosec} 2\alpha=\dfrac{\cos 2\alpha}{\sin 2\alpha}-\dfrac{1}{\sin 2\alpha}=\dfrac{\cos 2\alpha-1}{\sin 2\alpha}=\dfrac{-(1-\cos 2\alpha)}{\sin 2\alpha}$

$=\dfrac{-2\sin^2\alpha}{2\sin\alpha\cos\alpha}=-\dfrac{\sin\alpha}{\cos\alpha}=-\tan\alpha$. Similarly it will be found that $\cot 2\alpha+\operatorname{cosec} 2\alpha=\cot\alpha$. Substitute in (2) and (3).]

3. (*i*) $\tan^{-1}\frac{3}{5}$. (*ii*) $\tan^{-1}\frac{3}{4}$.

[**Hint.** (*i*) Factorise.
Or, solve the equation as a quadratic in x (or y). It will be found that the equation represents the two st. lines : $x-y=0$, and $x-4y=0$.]

Pages 89-90. 2. (*i*) $x^2+6xy-y^2=0$. (*ii*) $x^2-y^2=0$. (*iii*) $x^2-y^2=0$.

[**Hint.** (*ii*) It will be found that the equation of the bisectors is $\dfrac{x^2-y^2}{0}=\dfrac{xy}{-\sec\theta}$. Cross-multiply. (Compare the Note in Hint to Ex. 3, (*ii*), page 41.)]

3. [**Hint.** It will be found that the equation of the bisectors of the angles between the first pair of lines : $x^2-2pxy-y^2=0$ is
$px^2+2xy-py^2=0$ ...(1)

If it is the same as the equation of the second pair of lines :
$x^2-2qxy-y^2=0$ ...(2), then comparing coeffs. in (1) and (2), $\dfrac{p}{1}=\dfrac{2}{-2q}=\dfrac{-p}{-1}$.
The *symmetry* of the result $pq=-1$, shows that the first pair of lines bisects the angle between the second pair.]

5. [**Hint.** Find the equation of the bisectors of the angles between the second pair of lines, and show that it is the same as that of the bisectors of the angles between the first pair. Proceed as in Ex. 4.]

Pages 94-96. 1. (*i*) $(3, 2)$; $\tan^{-1}\frac{3}{5}$. (*ii*) $\left(\frac{2}{5}, -\frac{4}{5}\right)$; $45°$.

2. (*i*) $\left(-\frac{3}{2}, -\frac{5}{2}\right)$; $90°$. (*ii*) $(3, 2)$; $90°$. 3. $\dfrac{6}{\sqrt{10}}$.

4. (*i*) $-\frac{5}{2}, -\frac{10}{3}$.

(*ii*) 1. [**Hint.** It will be found that $k=0$, or 1. If $k=0$, the equation becomes $-4x+3y-12=0$ which, being of the first degree in x and y, represents *one* st. line. $\therefore$ $k=0$ is rejected.] (*iii*) -3.

5. (*i*) $c=0$, or $a+b=0$. (*ii*) $d=0$, or $bc-ad=0$.

[**Hint.** (*i*) The equation is $ax^2+by^2+cx+cy=0$ ...(1)
Comparing this with $ax^2+2hxy+by^2+2gx+2fy+c=0$,

here $a=a$, $h=0$, $b=b$, $g=\frac{1}{2}c$, $f=\frac{1}{2}c$, "c"$=0$*.
If (1) represents two st. lines, then "$abc+2fgh-af^2-bg^2-ch^2$"$=0$, etc.]

6. (*a*) $\tan^{-1}\dfrac{2\sqrt{h^2-ab}}{a+b}$, where the lines are represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$. (*b*) 2.

7. 7 ; imaginary.

Pages 97-98. Art. 53. 2. $x^2(c^2-a^2m^2)+2a^2mxy+y^2(c^2-a^2)=0$.

6. [**Hint.** It will be found that the equation of the lines joining the origin to the pts. of intersection of the line $x-y-1=0$ and the curve $5x^2+12xy-6y^2+4x-2y+3=0$ is $12x^2-y^2=0$, or $y^2=12x^2$, or $y=\pm 2\sqrt{3}x$, *i.e.*, $y=2\sqrt{3}x$, and $y=-2\sqrt{3}x$. Here $m_1=2\sqrt{3}$, $m_2=-2\sqrt{3}$, $\therefore\ m_1=-m_2$, $\therefore$ these lines make equal angles with the x-axis (Ex. 5).
$\therefore$ also they make equal angles with the y-axis.]

MISCELLANEOUS EXAMPLES ON CHAPTER IV

Pages 98-99. 1. $\dfrac{27}{4}$.

4. $x^2-3xy+2y^2=0$. [**Hint.** Proceed as in Ex. 2, Art. 49.]
5. [**Hint.** Proceed as in Ex. 4, Art. 49.]
6. [**Hint.** The equation is $\lambda y^2+(1-\lambda^2)xy-\lambda x^2=0$, or
$\lambda y^2+xy-\lambda^2 xy-\lambda x^2=0$. Factorise.]
7. The equation of the first pair of lines is $y^2-4xy-x^2=0$, or $x^2+4xy-y^2=0$ ...(1) [Form $ax^2+2hxy+by^2=0$]
and that of the second pair is $y^2+xy-x^2=0$, or $x^2-xy-y^2=0$ ...(2)
Find the equation of the bisectors of the angles between the lines (1), and show that it is the same as (2). Similarly find the equation of the bisectors of the angles between the lines (2), and show that it is the same as (1).]

8. [**Hint.** Proceed as in Ex. 4, Art. 51.] **9.** $\tan^{-1}\dfrac{3}{5}$.

10. $2x-3y+4=0$, $3x+2y+2=0$; $90°$.

11. $y-2x+1=0$, and $y-2x+2=0$. **12.** $\dfrac{7}{2\sqrt{5}}$. **13.** -1, or $-\dfrac{7}{2}$.

14. 2 ; $2x-6y-1=0$, and $x-2y+3=0$.

15. $2x-5y+2=0$, and $3x+2y-3=0$; $\dfrac{19}{4}$; $\left(\dfrac{11}{19}, \dfrac{12}{19}\right)$.

16. $2x^2-5xy+2y^2=0$. **17.** $xy=0$. **18.** $3x^2-2xy-y^2=0$. **20.** ± 1.

CHAPTER V

Page 101. 1. (*a*) $(x-a)^2+(y-b)^2=r^2$.
2. (*i*) $x^2+y^2-2x+4y-4=0$. (*ii*) $x^2+y^2-2ax-2by=0$.
3. (*i*) $x^2+y^2-4x-6y-12=0$. (*ii*) $x^2+y^2-4x-6y-12=0$.
[**Hint.** (*i*) The centre of the circle is (2, 3), and the radius=the distance between (2, 3) and (5, 7)=? $\therefore$ the equation of the circle is ?

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*Why to use inverted commas " ". In order to avoid confusion between c of the condition $abc+2fgh-af^2-bg^2-ch^2=0$ ...(1)
and c of the given equation $ax^2+by^2+cx+cy=0$ ...(2)
we put c of the condition (1) within inverted commas.

Or thus : Let the required equation of the circle be

$$(x-2)^2+(y-3)^2=a^2 \quad \ldots(1)$$

$\because$ it passes thro' (5, 7), $\therefore$? $\therefore$ a^2=?
Substitute this value of a^2 in (1).]

Pages 103-104. Art. 57. 1. (*b*) $\sqrt{2}$; (4, 8).

2. (*a*) $\sqrt{g^2+f^2-c}$; $(-g, -f)$. (*b*) (*i*) $\left(\frac{2}{7}, \frac{1}{14}\right)$; $\frac{\sqrt{101}}{14}$.

(*ii*) $\left(\frac{b}{2a}, \frac{c}{2a}\right)$; $\frac{\sqrt{b^2+c^2}}{2a}$. **3.** (*ii*) $\frac{\sqrt{g^2+f^2-ac}}{a}$; $\left(-\frac{g}{a}, -\frac{f}{a}\right)$.

Page 104. Art. 58.

1. [**Hint.** Choose the axes as in Art. 8-II, Rule II.]

Page 105. 1. (*i*) $x^2+y^2-22x-4y+25=0$;

(*ii*) $x^2+y^2+\frac{a^2-b^2}{b}y-a^2=0$.

(*iii*) $x^2+y^2-ax-by=0$; $\left(\frac{a}{2}, \frac{b}{2}\right)$, $\frac{\sqrt{a^2+b^2}}{2}$.

2. $x^2+y^2+x-5y-2=0$; $\left(-\frac{1}{2}, \frac{5}{2}\right)$; $\frac{\sqrt{34}}{2}$.

4. $x^2+y^2-6x-8y+15=0$.

5. $h(x^2+y^2)-(h^2+k^2-a^2)x-a^2h=0$.

Pages 106-107. 1. (*a*) $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$, where (x_1, y_1), (x_2, y_2) are the extremities of the diameter.

2. (*a*) $(x-a)(x-c)+(y-b)(y-d)=0$. [**Hint.** Proceed as in Art. 60.]

(*b*) $x^2+y^2-x-2y+1=0$; $\frac{1}{2}$; $(\frac{1}{2}, 1)$.

3. $x^2+y^2-3x-4y=0$. **4.** $x^2+y^2-ax-by=0$.

Page 111. 1. (*i*) $6x+8y+25=0$; (*ii*) $x-4y+9=0$. **2.** $c=0$.

3. $x+3y-10=0$, $x-3y-10=0$.

[**Hint.** In the equation of the circle $x^2+y^2=10$, put $x=1$, then $1+y^2=10$ or $y^2=9$, $\therefore$ $y=\pm 3$, $\therefore$ the pts. are (1, 3) and (1, −3).]

4. (*i*) $xx'+yy'+g(x+x')+f(y+y')+c=0$.

Page 113. 1. (*i*) $4x-3y=0$; (*ii*) $4x+y+2=0$.

2. (*i*) Tangent : $\frac{x}{a}+\frac{y}{b}=1$; Normal : $ax-by=0$.

(*ii*) Tangent : $gx+fy=0$; Normal : $fx-gy=0$.

3. $b=ma+c$. [**Hint.** If the line is a normal to the circle, it passes thro' the centre of the circle (Art. 66, Cor.).]

Pages 114-115. 1. (0, 5), (−4, 3).

2. $(0, a), \left[-\frac{2am}{1+m^2}, \frac{a(1-m^2)}{1+m^2}\right]$.

3. The line touches the circle at (−1, −1).

Page 117. 1. $\left(\frac{1}{5}, \frac{7}{5}\right)$, $(-1, -1)$; $\frac{6}{\sqrt{5}}$.

[**Hint.** Length of the chord (intercept)

=distance between the pts. of intersection.]

2. $2\sqrt{r^2-\frac{a^2b^2}{a^2+b^2}}$. 3. $2\sqrt{a^2-\frac{q^2}{1+p^2}}$; $\left(-\frac{pq}{1+p^2}, \frac{q}{1+p^2}\right)$.

Pages 119-121. 1. $c=\pm a\sqrt{1+m^2}$. 3. (7, 3), (2, 8).

4. (*i*) $c=-mh+k\pm a\sqrt{1+m^2}$. (*ii*) $C=-Aa-Bb\pm c\sqrt{A^2+B^2}$.

[**Hint.** Use the *tangent property* : **If a line touches a circle, the length of the ⊥ from the centre on the line=the radius.**]

5. $\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$. 6. $\frac{5}{4}$ or $-\frac{45}{4}$.

7. $x^2+y^2-2x+4y-3=0$.

[**Hint.** Radius of the circle=length of the ⊥ from the centre on the line.]

10. $c=\pm r\sqrt{1+m^2}$; $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$.

11. $\sqrt{3}x-y+4=0$, $\sqrt{3}x-y-4=0$.

12. (*i*) $x+2y-2\sqrt{5}=0$, $x+2y+2\sqrt{5}=0$.

(*ii*) $x=\pm 2$. [**Hint.** Use Method II of solved Ex. 2, Art. 71.]

13. $3x-4y+20=0$, $3x-4y-10=0$.

[**Hint.** Use Method II of solved Ex. 2, Art. 71.]

14. (*i*) $x+y-3\sqrt{2}=0$, $x+y+3\sqrt{2}=0$.

(*ii*) $4x+3y+5=0$, $4x+3y-25=0$.

Page 122. **Ex.** Outside.

Page 124. **Ex.** $y-2=0$, $4x-3y-10=0$; (0, 2), $\left(\frac{8}{5}, -\frac{6}{5}\right)$.

[**Hint.** To find the co-ordinates of the pts. of contact proceed as in Ex. 8, Art. 71.]

Page 126. 1. $hx+ky=r^2$.

2. (3, 4), (4, −3) ; $3x+4y-25=0$, $4x-3y-25=0$.

[**Hint.** Find the equation of the chord of contact of tangents from (7, 1) and then its pts. of intersection with the circle.]

Pages 127-129. 3. (*i*) $2x-4y-3=0$. (*ii*) $x-1=0$.

4. (*i*) $ax-by-a^2=0$. (*ii*) $4x+4y-37=0$.

Page 130. 1. (20, −5). 3. (*i*) (−2, 1). (*ii*) ($-1, \frac{1}{2}$).

Page 132. 1. [**Hint.** Let the equation of the circle be $x^2+y^2=a^2$, and $(a\cos\theta_1, a\sin\theta_1)$, $(a\cos\theta_2, a\sin\theta_2)$ the ends of the chord. Find the slope of the chord (Art. 29, Cor.), also the slope of the line passing thro' the centre and the mid-pt. of the chord.]

2. [**Hint.** *Any* pt. on the circle $x^2+y^2=a^2$ is $(a\cos\theta, a\sin\theta)$.]

3. [**Hint.** Eliminate θ by squaring the two equations and adding.]

Pages 135-136. 3. $2x-3y+13=0$; $\left(-\frac{32}{13}, \frac{48}{13}\right)$.

Pages 137-138.

1. $\sqrt{x_1^2+y_1^2-a^2}$. [**Hint..** Proceed as in Art. 85] ; $\frac{2}{\sqrt{3}}$.

2. (*i*) 3 ; (*ii*) $\sqrt{\frac{a(x'^2+y'^2)+2gx'+2fy'+c}{a}}$.

4. [**Hint.** Let (x_1, y_1) be *any* pt.]

5. $\left(\frac{33}{4}, 2\right)$; $\frac{1}{4}$.

[**Hint.** Let (x_1, y_1) be the pt. the tangents from which to the three circles are equal. Then

$\sqrt{x_1^2+y_1^2-16x_1+60}=\sqrt{x_1^2+y_1^2-12x_1+27}=\sqrt{x_1^2+y_1^2-16x_1-12y_1+84}$.

Square, and cancel $x_1^2+y_1^2$ from each member. Solve for x_1, y_1.]

Page 141.

1. (*i*) $2x-3y-4=0$. (*ii*) $5x-3y+5=0$. (*iii*) $ax+by+\frac{ac}{a-b}=0$.

2. $x-y=0$; $\sqrt{\frac{(a+b)^2-8c}{2}}$.

[**Hint.** (*ii*) Find the length of the intercept made by the first circle on the radical axis (common chord) by the method of Art. 70. (Use the intercept formula.)]

3. $2\sqrt{2}$. **4.** (4, 3), (-3, -4).

5. [**Hint.** (*i*) Find the equation of the radical axis of the two circles. (*ii*) Find the condition that this radical axis may touch one of the two circles (say the first). Use the method of intersections.]

6. (1, 1).

7. $3x+4y-1=0$.

[**Hint.** The radical axis of two touching circles is their common tangent at the pt. of contact.]

8. $\left(\frac{1}{5}, \frac{3}{5}\right)$, $4x-3y+1=0$, external.

[**Hint.** Use Method II of solved Ex. 2, Art. 87.

Or thus : Hint. Use Method I of solved Ex. 2, Art. 87. *The contact is external or internal according as the centres of the circles lie on opposite sides or on the same side of the common tangent.* (Use Rule of Art. 45.)]

Page 142. **2.** $4x+y-1=0$. **3.** $\sqrt{\frac{249}{5}}$.

Page 143. **1.** (*i*) (2, 1). (*ii*) (2, -1). (*iii*) $\left(\frac{9}{8}, -\frac{5}{8}\right)$.

Page 145.

1. $2aa'+2bb'=c+\lambda c'$.

3. $x^2+y^2+6x-3y=0$.

[**Hint.** Let the required equation of the circle be

$x^2+y^2+2gx+2fy+c=0$ ... (1)

$\because$ it passes thro' (0, 0), $\therefore$ $c=0$ ... (2)

$\because$ it cuts $x^2+y^2-8y+12=0$ orthogonally,

$\therefore$ $2g(0)+2f(-4)=c+12$ ... (3) $[2g_1g_2+2f_1f_2=c_1+c_2]$

Again $\because$ it cuts $x^2+y^2-4x-6y-3=0$ orthogonally,

$\therefore$ $2g(-2)+2f(-3)=c-3$ ... (4)

Solve (2), (3), (4) for g, f, c, and substitute these values in (1).]

4. $7x^2+7y^2-8x-8y-12=0$.

(*ii*) (*a*) $\frac{8}{3}$, (*b*) $\left(\frac{2}{3}, 0\right)$, (*c*) $3x+2=0$;

(*iii*) (*a*) 4, (*b*) $(-1, 0)$, (*c*) $x-1=0$.

[**Hints.** (*i*) Compare the equation with $y^2=4ax$. Here $4a=8$, $\therefore$ $a=2$.
(*ii*) Divide thro' out by 3 (to reduce the equation to the form $y^2=4ax$).
(*iii*) (*a*) Use Art. 98, **N.B.**

(*b*) and (*c*) Draw a rough Fig. (Art. 96, Ex. 1), and proceed as in Cor. of that Ex.]

2. $\frac{4}{3}$, $(\frac{1}{3}, 0)$.

3. (*i*) (*a*) $\frac{4}{3}$, (*b*) $(0, \frac{1}{3})$, (*c*) $3y+1=0$;

(*ii*) (*a*) 12, (*b*) $(0, -3)$, (*c*) $y-3=0$.

4. $4x-1=0$; (*i*) $(\frac{1}{4}, \frac{1}{2})$, (*ii*) $(\frac{1}{4}, -\frac{1}{2})$.

Pages 159-160. 1. $3x+2y+3=0$.

Page 161. 1. (*i*) $y=mx+\frac{a}{m}$;

(*ii*) $y=mx-2am-am^3$, m is the slope of the normal.

2. Tangent : $3x-4y+8=0$, Normal : $12x+9y-68=0$.

3. Tangent : $x-y+2=0$, Normal : $x+y-6=0$.

4. At $(a, 2a)$, Tangent : $x-y+a=0$, Normal : $x+y-3a=0$;
at $(a, -2a)$, Tangent : $x+y+a=0$, Normal : $x-y-3a=0$.

5. At (3, 6), Tangent : $x-y+3=0$, Normal : $x+y-9=0$;
at (3, −6), Tangent : $x+y+3=0$, Normal : $x-y-9=0$.

[**Hint.** Here $4a=12$, $\therefore$ $a=3$, $\therefore$ the ends of the latus rectum are $(a, 2a)$, $(a, -2a)$, *i.e.*, (3, 6), (3, −6).]

Page 162.

Ex. (*a*) $y=mx-2am-am^3$, where m is the slope of the normal.

(*b*) (*i*) $\sqrt{3}x-y-5a\sqrt{3}=0$, (*ii*) $\left(\frac{a}{3}, -\frac{2a}{\sqrt{3}}\right)$.

Page 163. 3. $y^2=a(x-a)$.

[**Hint.** If two normals make complementary angles θ, $90°-\theta$ with the x-axis, then $m_1m_2=\tan\theta\tan(90°-\theta)=\tan\theta\cot\theta=1$. $\therefore$ $m_1m_2=1$.]

Page 164. 1. $(a, 2a)$, $\left(\frac{a}{9}, -\frac{2a}{3}\right)$. 2. (9, 3), (4, 2).

Page 165. 1. $24\sqrt{2}$. [**Hint.** First find the equation of the normal at (3, 6) to the parabola, and then use the Note given in Art. 70.]

Pages 167-168.

1. $\left(\frac{a}{4}, a\right)$. [**Hint.** Proceed as in Ex. 2, Art. 71.]

2. $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. 3. (*ii*) 8 ; (2, −4).

4. $am^2=nl$; $\left(\frac{n}{l}, -\frac{2am}{l}\right)$. [**Hint.** Proceed as in Ex. 9, Art. 71.]

6. (*a*) $x-y+a=0$, $x+y+a=0$. (*b*) $\left(\frac{4}{3}, \frac{8}{\sqrt{3}}\right)$.

7. (*i*) $(3a, 2\sqrt{3}a)$. (*ii*) $\left(\frac{a}{3}, -\frac{2a}{\sqrt{3}}\right)$.

8. $x+y+a=0$. [**Hint.** See Note in Misc. Ex. 19, (*b*), Chap. III.]

Note. If the tangent cuts off intercepts *equal in magnitude but opposite in sign*, its equation is $x-y+a=0$.

9. Tangent : $9x-12y+64=0$, pt. of contact $\left(\frac{64}{9}, \frac{32}{3}\right)$;

Tangent : $4x+3y+9=0$, pt. of contact $\left(\frac{9}{4}, -6\right)$.

[**Hint.** (*i*) Here $4a=16$, $\therefore$ $a=4$, $m=\frac{3}{4}$.

(*ii*) Here $4a=16$, $\therefore$ $a=4$, $m=-\frac{4}{3}$.]

Page 169. Ex. (*i*) Inside ; (*ii*) outside ; (*iii*) on.

[**Hint.** (*iii*) Divide thro' out by 2 (to reduce the equation to the form $y^2=4ax$). It becomes $y^2=\frac{9}{2}x$.]

Page 170. 1. $9x-4y+4=0$, $x-4y+36=0$. **2.** $\tan^{-1}\frac{1}{2}$.

Page 171. Art. 109.

Ex. (*i*) $y=kx$; (*ii*) $x=\frac{a}{k}$; (*iii*) $kx^2-y^2+2ax=0$; (*iv*) $y=ak$.

[**Hint.** Proceed as in Art. 109, and get equation (1).

$$m_1+m_2=\frac{y_1}{x_1}, \quad m_1m_2=\frac{a}{x_1} \quad \ldots(2)$$

(*i*) $m_1+m_2=k$. Substitute from (2).

(*iii*) $m_1^2+m_2^2=k$ or $(m_1+m_2)^2-2m_1m_2=k$. Substitute from (2).

(*iv*) $\frac{1}{m_1}+\frac{1}{m_2}=k$ or $\frac{m_1+m_2}{m_1m_2}=k$. Substitute from (2).]

Page 171. Art. 110.

2. [**Hint.** The pt. of intersection of two $\perp$ tangents to the parabola $y^2=4ax$ lies on the directrix $x=-a$ (Art. 109).
$\therefore$ let the pt. be $(-a, y_1)$. Find the equation of its chord of contact.]

3. (4, 4), (1, 2). [**Hint.** Proceed as in Ex. 2, Art. 77.]

Page 172. 1. (*a*) [See Art. 78.]

Page 173. 1. $(-a, 3a)$. **2.** $(0, \frac{1}{2})$. **3.** $\left(\frac{C}{A}, -2\frac{B}{A}a\right)$.

Page 174. Art. 114.

2. [**Hint.** Let A(0, 0) be the vertex, and P(at^2, $2at$) *any* pt. on the parabola. Then the co-ordinates of the mid-pt. of AP are.

$$x=\frac{0+at^2}{2}=\frac{at^2}{2} \quad \ldots(1)$$

$$y=\frac{0+2at}{2}=at \quad \ldots(2).$$ Eliminate t from (1) and (2).

****Or thus :** The equation of the chord, whose mid-pt. is (x_1, y_1), is $T=S_1$ (see N.B. in solved Ex. 1, Art. 84), *i.e.*, $yy_1-2a(x+x_1)=y_1^2-4ax_1$ or $yy_1-2ax=y_1^2-2ax_1$. If it passes thro' the vertex A(0, 0), then $0=y_1^2-2ax_1$, $\therefore$ the locus of (x_1, y_1) is [changing (x_1, y_1) to (x, y)], $0=y^2-2ax$.]

3. $y^2=\frac{4a}{9}x$. [**Hint.** Let P be $(at^2, 2at)$. Then Q is $(at^2, -2at)$.]

5. (*ii*) m is the slope of the tangent. (*iii*) $y=mx+\frac{a}{m}$.

Pages 174-175. Art. 115. 1. (*ii*) $tx+y-2at-at^3=0$.

2. $[at_1t_2, a(t_1+t_2)]$.

3. $c=-2am-am^3$.

[**Hint.** It will be found that the equation of the normal at $(at^2, 2at)$ is

$$tx+y-2at-at^3=0 \quad ...(1)$$

If it is the same as the equation of the given line

$$mx-y+c=0 \quad ...(2)$$

then compare coeffs. in (1) and (2), and eliminate t between the two resulting equations.]

Page 176. Ex. $x^3+y^2(x+a)=0$.

MISCELLANEOUS EXAMPLES ON CHAPTER VI

Pages 178-183.

1. $a^2x^2-2abxy+b^2y^2-2a^3x-2b^3y+a^4+a^2b^2+b^4=0$.

2. (*i*) $x^2+2xy+y^2-8x+8y-16=0$.

3. $x^2+2xy+y^2+2x-10y-5=0$; $3\sqrt{2}$; $x+y-2=0$; $\left(\frac{7}{4}, \frac{1}{4}\right)$.

5. (*i*) $(-2, -3)$, $\left(-\frac{3}{2}, -3\right)$, 2, $2x+5=0$.

(*ii*) $(-1, 0)$, $(-\frac{1}{2}, 0)$, 2, $2x+3=0$.

6. $(1, 2)$, $(0, 2)$, 4 ; $y-2=0$, $x-2=0$.

7. (*i*) $\left(-\frac{5}{2}, \frac{3}{2}\right)$, $2y-3=0$, $\left(-\frac{13}{4}, \frac{3}{2}\right)$, 3.

(*ii*) $\left(a, \frac{a}{2}\right)$, $x-a=0$, $(a, 0)$, $2a$.

8. (*i*) $2x+y=0$. (*ii*) $y-6=m(x-12)$ or $y+6=m(x-12)$.

10. (*i*) $y^2=4ax$ is a parabola whose vertex is the origin, latus rectum $=4a$, and axis in the +ve direction of the x-axis ; $x^2=4by$ is a parabola whose vertex is the origin, latus rectum $=4b$, and axis in the +ve direction of the y-axis. (*iii*) $a^{\frac{1}{3}}x-b^{\frac{1}{3}}y=0$.

11. $xx_1=2a(y+y_1)$. 12. $\tan^{-1}\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}$.

13. 1, 2, −3 ; $x-y-3=0$, $2x-y-12=0$, $3x+y-33=0$.

15. $\frac{80}{9}$. 17. $(2, -3)$. 18. $(at^2, 2at)$. 19. (*ii*) (1,1).

20. $a\sin^2\alpha=-p\cos\alpha$; $\left(-\frac{p}{\cos\alpha}, -\frac{2a\sin\alpha}{\cos\alpha}\right)$.

21. $a^{\frac{1}{3}}x+b^{\frac{1}{3}}y+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$.

22. [**Hint.** The equation of *any* tangent to the parabola $y^2=8x$, in the m-form, is $y=mx+\frac{2}{m}$. $\left[y=mx+\frac{a}{m}\right]$

If it touches the circle, use the tangent property.]

23. $9x+3y+1=0$, $\left(\frac{1}{9}, -\frac{2}{3}\right)$; $x-3y+9=0$, $(9, 6)$.

[**Hint.** Let m be the slope of the tangent.
$\because$ it makes an angle of 45° with $y=2x+3$,
$\therefore \tan 45° = \pm\frac{m-2}{1+m(2)}$. Find the values of m, and write down the equation of the tangent using $y=mx+\frac{a}{m}$.]

24. $x-y+1=0$, $x-4y+16=0$. **26.** $y^2-4ax=(a+x)^2\tan^2\alpha$.
27. $x^2-y^2+6ax+a^2=0$, where $y^2=4ax$ is the equation of the parabola.
28. (*i*) $\left[\frac{a}{mm'}, a\left(\frac{1}{m}+\frac{1}{m'}\right)\right]$.

[**Hint.** (*ii*) The co-ordinates of the pt. of intersection are

$$x=\frac{a}{mm'} \quad ...(1), \qquad y=a\left(\frac{1}{m}+\frac{1}{m'}\right) \quad ...(2)$$

If $mm'=c$, then from (1), $x=\frac{a}{c}$, which is the required locus of the intersection.]

29. 45°. **30.** $\left(-\frac{p}{\cos\alpha}, -\frac{2a\sin\alpha}{\cos\alpha}\right)$.

33. [**Hint.** The equation of the tangent at $(at^2, 2at)$ is $ty=x+at^2$ ...(1), and the equation of the tangent at $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ is $-\frac{y}{t}=x+\frac{a}{t^2}$ ...(2)
(*i*) Show that (1) is $\perp$ to (2).
(*ii*) For the pt. of intersection, find the value of x from (1) and (2), and show that $x=-a$.]

34. [**Hint.** Let the ends of a focal chord be $(at^2, 2at)$, $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.
Then the co-ordinates of the mid-pt. of the chord are $x=\frac{a}{2}\left(t^2+\frac{1}{t^2}\right)$...(1)
$y=a\left(t-\frac{1}{t}\right)$ ...(2). Eliminate t from (1) and (2) [by squaring (2) and substituting the value of $t^2+\frac{1}{t^2}$ from (1)].]

****Short-cut.** The equation of the chord, whose mid-pt. is (x_1, y_1), is $T=S_1$ (see N.B. in solved Ex. 1, Art. 84), *i.e.*, $yy_1-2a(x+x_1)=y_1^2-4ax_1$ or $yy_1-2ax=y_1^2-2ax_1$.
If it passes thro' the focus $(a, 0)$, then ? Find the locus of (x_1, y_1).]

35. [**Hint.** Let O be the pt. $(k, 0)$, and P, Q the pts. $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$. Then the chord PQ, $y(t_1+t_2)=2x+2at_1t_2$ passes thro' $(k, 0)$, $\therefore\ 0=2k+2at_1t_2$.]

37. [**Hint.** The normal at $(at_1^2, 2at_1)$ passes thro' $(at_2^2, 2at_2)$, $\therefore$? Cancel the common factor (t_2-t_1).]

38. $y^2=a(x-a)$.

40. $y-1=0$. [**Hint.** Here $4a=6$, $\therefore\ a=\frac{3}{2}$, $m=3$. Use $y=\frac{2a}{m}$.]

CHAPTER VII

Page 187. **1.** (*i*) $4x^2+9y^2-36=0$. (*ii*) 1 ; 2, 1.

2. (*i*) $3x^2+4y^2-48=0$. (*ii*) $3x^2+5y^2-32=0$; $\sqrt{\frac{2}{5}}$.

[**Hint.** (*ii*) Let the required equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

$\because$ it passes thro' (−3, 1), (2, −2),

$\therefore \frac{9}{a^2}+\frac{1}{b^2}=1$ ...(2)

$\frac{4}{a^2}+\frac{4}{b^2}=1$ ...(3)

From (2) and (3), find the values of $\left(\frac{1}{a^2}, \frac{1}{b^2}\right.$ and $\therefore$ of$\left.\right)$ a^2, b^2, and substitute these values in (1). For eccentricity use $b^2=a^2(1-e^2)$.]

3. $3x^2+5y^2-32=0$. **4.** $91x^2-24xy+84y^2-170x-360y+475=0$.

Pages 189-190. **2.** $8x^2+9y^2-288=0$; $x=\pm 18$.

Pages 192-193. **1.** $\frac{1}{3}$.

[**Hint.** $2\frac{b^2}{a}=4$ ...(1), SA=CA−CS=$a-ae=\frac{3}{2}$ (Given) ...(2)

From (1), $\frac{2a^2(1-e^2)}{a}=4$ or $a(1-e^2)=2$ ...(3)

From (2), $a(1-e)=\frac{3}{2}$ ...(4). Divide (3) by (4).]

2. $x^2+2y^2-4=0$.

[**Hint.** $2b=2ae$ or, squaring, $b^2=a^2e^2$, *i.e.*, $a^2(1-e^2)=a^2e^2$, $\therefore$ $e^2=?$

Also $2\frac{b^2}{a}=2$, *i.e.*, $\frac{a^2(1-e^2)}{a}=1$. Substitute the value of e found above.

$\therefore$ $a=?$ $b^2=a^2(1-e^2)=?$]

3. $20x^2+36y^2-405=0$.

4. (*a*) (*i*) $\frac{1}{\sqrt{3}}$, $\left(\pm\frac{1}{\sqrt{6}}, 0\right)$, $\frac{2\sqrt{2}}{3}$. (*ii*) $\frac{1}{2}$, $\left(0, \pm\frac{1}{2\sqrt{3}}\right)$, $\frac{\sqrt{3}}{2}$.

(*b*) $\frac{5}{2}$, $\frac{5}{3}$, $\frac{\sqrt{5}}{3}$, $\left(\pm\frac{5\sqrt{5}}{6}, 0\right)$. **5.** $\frac{1}{2}$, 2, 3.

6. (*i*) $\frac{\sqrt{3}}{2}$. (*ii*) $\frac{1}{\sqrt{2}}$.

[**Hint.** (*i*) $\frac{2b^2}{a}=\frac{1}{2}.2b=b$, $\therefore$ $2\frac{b}{a}=1$ or, squaring, $\frac{4b^2}{a^2}=1$,

i.e., $\frac{4a^2(1-e^2)}{a^2}=1$.]

7. $\frac{\sqrt{5}-1}{2}$.

[**Hint.** It will be found that $e=\frac{-1\pm\sqrt{5}}{2}$.

$\therefore$ (1) $e=\frac{\sqrt{5}-1}{2}$. (2) $e=-\frac{\sqrt{5}+1}{2}$ which, being numerically >1, is

impossible ($\because$ eccentricity of an ellipse is <1).]

Pages 194-195. 1. $2x+\sqrt{3}y-4\sqrt{3}=0$.

2. [**Hint.** (*i*) The extremities of the major axis are A$(-a, 0)$, A$'(a, 0)$.
(*ii*) The extremities of the minor axis are B$(0, b)$, B$'(0, -b)$.]

3. (*i*) At $\left(-ae, \frac{b^2}{a}\right)$, $ex-y+a=0$;

at $\left(-ae, -\frac{b^2}{a}\right)$, $ex+y+a=0$;

at $\left(ae, \frac{b^2}{a}\right)$, $ex+y-a=0$; at $\left(ae, -\frac{b^2}{a}\right)$, $ex-y-a=0$.

Page 196.

1. At (3, 2), Tangent : $3x+8y-25=0$, Normal : $8x-3y-18=0$;
at $(-3, 2)$, Tangent : $3x-8y+25=0$, Normal : $8x+3y+18=0$.

2. Tangent : $ex+y-a=0$, Normal : $x-ey-ae^3=0$.
[**Hint.** In the equation of the normal obtained, put $b^2=a^2(1-e^2)$.]

3. At $\left(-1, \frac{2}{\sqrt{3}}\right)$, Tangent : $x-\sqrt{3}y+3=0$, Normal : $3x+\sqrt{3}y+1=0$;

at $\left(-1, -\frac{2}{\sqrt{3}}\right)$, Tangent : $x+\sqrt{3}y+3=0$, Normal : $3x-\sqrt{3}y+1=0$;

at $\left(1, \frac{2}{\sqrt{3}}\right)$, Tangent : $x+\sqrt{3}y-3=0$, Normal : $3x-\sqrt{3}y-1=0$;

at $\left(1, -\frac{2}{\sqrt{3}}\right)$, Tangent : $x-\sqrt{3}y-3=0$, Normal : $3x+\sqrt{3}y-1=0$.

Page 197. Ex. $\left(2\sqrt{2}, \frac{5}{\sqrt{2}}\right)$, $\left(-2\sqrt{2}, -\frac{5}{\sqrt{2}}\right)$.

Page 198. Ex. [**Hint.** First find the pts. of intersection, and then the distance between them.]

Pages 199-200. 1. (*ii*) $c=\pm\sqrt{a^2m^2+b^2}$. (*iii*) $c=\pm r\sqrt{1+m^2}$.

[**Hint.** (*iii*) The equation of the circle is $\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$,

$\therefore$ in the condition of tangency for the ellipse, put $a^2=r^2$, $b^2=r^2$.]

2. $\left(-\sqrt{\frac{3}{10}}, \sqrt{\frac{2}{15}}\right)$. 5. $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$.

6. $a^2l^2+b^2m^2=n^2$; $\left(-\frac{l}{n}a^2, -\frac{m}{n}b^2\right)$.

[**Hint.** Proceed as in Ex. 9, Art. 71.]

7. $y=-x\pm\sqrt{a^2+b^2}$.

Note. If the intercepts are *equal in magnitude but opposite in sign*, the equations of the tangents are $y=x\pm\sqrt{a^2+b^2}$.

[**Hint.** See Note in Misc. Ex. 19, (*b*), Chap. III.]

8. (*i*) $4x-y\pm7=0$. (*ii*) $4x+y\pm7=0$.

Page 201. 1. No.

2. Without. [**Hint.** Divide thro' out by 140 (to reduce the equation to the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$).]

Page 202. **1.** $4x-5y+25=0$, $x-4y-13=0$.
2. $x-y-3=0$, $x+5y-9=0$. **3.** $5x+2y-9=0$.

Page 203. **1.** (*a*) The locus of the pt. of intersection of two ⊥ tangents to an ellipse is a circle called the **director circle** of the ellipse. (*b*) $x^2+y^2=a^2+b^2$. **2.** (*i*) $kx^2-2xy-ka^2=0$; (*ii*) $kx^2-y^2-ka^2+b^2=0$; (*iii*) $k^2y^2-2xy-k^2b^2=0$; (*iv*) $x^2-2xy\cot 2\alpha-y^2-a^2+b^2=0$.

[**Hint.** Proceed as in Art. 129, and get equation (1).

$$m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}, \quad m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2} \quad \ldots(2)$$

(*i*) $m_1+m_2=k$. Substitute from (2).
(*ii*) $m_1m_2=k$. Substitute from (2).
(*iii*) $\frac{1}{m_1}+\frac{1}{m_2}=k^2$ or $\frac{m_1+m_2}{m_1m_2}=k^2$. Substitute from (2).
(*iv*) $\theta_1+\theta_2=2\alpha$, $\therefore$ $\tan(\theta_1+\theta_2)=\tan 2\alpha$
or $\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}=\tan 2\alpha$ or $\frac{m_1+m_2}{1-m_1m_2}=\tan 2\alpha$. Substitute from (2).]

Page 204. **1.** $3x+8y-5=0$.

Page 205. **1.** $\left(-\frac{m}{c}a^2, \frac{b^2}{c}\right)$. **2.** $\left(-\frac{4}{3}, 2\right)$.

Page 206. **Ex.** [**Hint.** Proceed as in Ex., Art. 81.]

Page 207. **1.** A variable in terms of which the co-ordinates of *any* point on a curve are expressed, is called a **parameter,** and the co-ordinates are called the **parametric co-ordinates.**
(*i*) $(a\cos\theta, a\sin\theta)$, (*ii*) $(at^2, 2at)$, (*iii*) $(a\cos\theta, b\sin\theta)$.
2. $(a\cos\theta, b\sin\theta)$.
3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{x}{a}=0$.

[**Hint.** Let A′ $(a, 0)$ be one (positive) end of the major axis, and P$(a\cos\theta, b\sin\theta)$ *any* pt. on the ellipse. Then the co-ordinates of the mid-pt. of A′P are $x=\frac{a+a\cos\theta}{2}$, $y=\frac{0+b\sin\theta}{2}$, $\therefore$ $\cos\theta=?$ $\sin\theta=?$
Eliminate θ by squaring and adding.
****Short-cut.** The equation of the chord, whose mid-pt. is (x_1, y_1), is $T=S_1$ (see N.B. in solved Ex. 1, Art. 84),

i. e., $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$ or $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$.

If it passes thro' A′$(a, 0)$, one (positive) end of the major axis, then ? Find the locus of (x_1, y_1).]

4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{y}{b}=0$.

Pages 208-210.

2. $\left(a\frac{\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}, b\frac{\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}\right)$. **3.** (*i*) [See Art. 134.]

6. (*i*) $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=\cos\beta$. (*ii*) $\frac{x}{a}+\frac{y}{b}=\frac{\sqrt{3}+1}{2}$.

[**Hint.** (*i*) Use Art. 136. (*ii*) $\cos\frac{\pi}{12}=\cos 15°=\cos(45°-30°)=$?]

Pages 211-212. 3. k, $\sqrt{k^2-h^2}$. [**Hint.** The focal distance of B$(0, b)=a+ex_1=a+e(0)=k$ (Given), also $2ae=2h$.]

Pages 217-219. Art. 137 (*g*).

2. $2x+3y=0$. [**Hint.** Use $y=-\frac{b^2}{a^2m}x$.]

7. (*i*) $b^2x-a^2y=0$. (*ii*) $bx+ay=0$. 8. $\frac{1}{\sqrt{3}}$.

MISCELLANEOUS EXAMPLES ON CHAPTER VII

Pages 219-224. 1. $27x^2+24xy+20y^2+72x-72y+72=0$.

2. $4\sqrt{2}, 2\sqrt{6}$; $3\sqrt{2}$; $(0, \pm 2\sqrt{2})$; $(0, \pm\sqrt{2})$; $y\pm 4\sqrt{2}=0$.

3. An ellipse whose major axis$=2\sqrt{3}$ and lies along the x-axis, and whose minor axis$=2\sqrt{2}$ and lies along the y-axis ;

$$\left(-1, \frac{2}{\sqrt{3}}\right), \left(-1, -\frac{2}{\sqrt{3}}\right); \left(1, \frac{2}{\sqrt{3}}\right), \left(1, -\frac{2}{\sqrt{3}}\right).$$

5. (*i*) $\frac{1}{2}$; (3, 1), (1, 1) ; $x-6=0$, $x+2=0$.

(*ii*) $\frac{1}{2}, \left(-1, 1\pm\frac{1}{2\sqrt{6}}\right)$; $\sqrt{6}y-\sqrt{6}\pm 2=0$.

6. (*i*) $\frac{1}{\sqrt{2}}, \frac{3}{2\sqrt{2}}, \left(\frac{3}{4}, \pm\frac{3}{4}\right)$.

[**Hint.** (*ii*) The extremities of the minor axis are $x'=\pm b$, $y'=0$.]

7. (3, 5) ; Equations : $x-3=0$, $y-5=0$; Lengths : 10, 6 ; (3, 9), (3, 1) ; $4y-45=0$, $4y+5=0$.

8. [**Hint.** Solve the equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $x^2+y^2=ab$.

It will be found that $x^3=\frac{a^2b}{a+b}$, $y^2=\frac{ab^2}{a+b}$.

$\therefore$ a pt. of intersection is $\left(\sqrt{\frac{a^2b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$.

It will be found that the slopes of the tangents to the ellipse and the circle at this pt. are $m_1=-\frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}$, $m_2=-\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ ...(1)

$\therefore$ if θ is the angle between the tangents, then

$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$. Substitute from (1).]

9. $\left(\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right); \left(-\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$.

10. $\left(\frac{16}{5}, \pm\frac{9}{5}\right); \left(-\frac{16}{5}, \pm\frac{9}{5}\right)$.

11. (5, 2), (−5, −2). **13.** 4.

14. $(3, 2\sqrt{3})$; $\left(-1, \frac{2}{\sqrt{3}}\right)$. **15.** (1, −1).

16. $ax-by\pm\sqrt{a^4+a^2b^2+b^4}=0$, $ax+by\pm\sqrt{a^4+a^2b^2+b^4}=0$.

17. (*iii*) $x-y\pm5=0$, $x+y\pm5=0$. **19.** $x^2+y^2=a^2+b^2$.

20. $y-3=0$, $x+y-5=0$; $3x+8y-24=0$.

21. [**Hint.** Draw a rough Fig. Let T be (x_1, y_1).]

22. (*i*) $x-y+10=0$, (*ii*) (−1, 2).

25. [**Hint.** The co-ordinates of the extremities of the chord are

$$(a\cos\phi_1, b\sin\phi_1), (a\cos\phi_2, b\sin\phi_2).$$

∴ the equation of the chord is

$$\frac{x}{a}\cos\frac{\phi_1+\phi_2}{2}+\frac{y}{b}\sin\frac{\phi_1+\phi_2}{2}=\cos\frac{\phi_1-\phi_2}{2}. \quad \text{(Art. 136)}$$

∵ it passes thro' the focus $(ae, 0)$, ∴ ?]

26. (*b*) $\dfrac{4x^2}{a^2(1+e^2)^2}+\dfrac{4y^2}{b^2}=1.$

[**Hint.** (*a*) Take the equation of the tangent at $(a\cos\theta, b\sin\theta)$.
(*b*) Take the equation of the normal at $(a\cos\theta, b\sin\theta)$.

29. $\dfrac{18}{\cos^2\alpha}+\dfrac{12}{\sin^2\alpha}=\dfrac{1}{p^2}$.

[**Hint.** The equation of the ellipse is $\dfrac{x^2}{\frac{1}{2}}+\dfrac{y^2}{\frac{1}{3}}=1.$

$\left(\text{Compare with } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.\right)$ Here $a^2=\frac{1}{2}$, $b^2=\frac{1}{3}$, ∴ $a=\dfrac{1}{\sqrt{2}}$, $b=\dfrac{1}{\sqrt{3}}$,

∴ the equation of the normal at $(a\cos\theta, b\sin\theta)$, *viz.*,

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \text{ becomes}$$

$$\frac{x}{\sqrt{2}\cos\theta}-\frac{y}{\sqrt{3}\sin\theta}=\tfrac{1}{2}-\tfrac{1}{3}=\tfrac{1}{6}.$$ Proceed as in Ex. 4, Art. 136.]

30. Equations : $x-8=0$, $9x-40y+72=0$; Lengths : $\dfrac{18}{5}$, $\dfrac{82}{5}$.

[**Hint.** It will be found that the foci are $(\pm8, 0)$. Find the equations and lengths of the joins of

(*i*) (8, 0) and $\left(8, \dfrac{18}{5}\right)$; (*ii*) (−8, 0) and $\left(8, \dfrac{18}{5}\right)$.

Or thus : To find the lengths. The focal distances are

$$a\pm ex_1, \textit{ i.e., } 10\pm\frac{4}{5}(8) \text{ or } \frac{82}{5}, \frac{18}{5}. \ ]$$

33. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, an ellipse. [**Hint.** Eliminate the variable (parameter) t (Misc. Ex. 23, Chap. III, Rule).]

34. [**Hint.** Subtangent $=\frac{a^2-x_1^2}{x_1}$ (As in Art. 137,(*b*), Cor.). (Here $x_1=3$)

$=\frac{a^2-9}{3}=\frac{16}{3}$ (Given), $\therefore\ a^2=$?

Also $\left(3, \frac{12}{5}\right)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\therefore$?

Substitute the value of a^2 found above. $b^2=$?]

CHAPTER VIII

Page 228.

1. (*a*) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $2a$, $2b$ are the transverse and conjugate axes.

(*b*) $9x^2-4y^2-9=0$.

2. $41x^2-4y^2-25=0$. **3.** $x^2-4xy+y^2+4x-4=0$.

Pages 229-231. Art. 140.

1. (*i*) $45x^2-36y^2-80=0$. (*ii*) $x^2-y^2-2=0$.

2. $9x^2-16y^2-36=0$. **3.** $\sqrt{\frac{5}{3}}$. **4.** $2\ ;\ \left(\pm\frac{4}{\sqrt{3}}, 0\right)$.

6. $(0, \pm\sqrt{13})$.

[**Hint.** It will be found that the equation is $\frac{y^2}{4}-\frac{x^2}{9}=1$.

[R.H.S.=1, and coeffs. of x^2, y^2 as denoms. (*Note this step*)]

Here (+ve denom.) $a^2=4$, (−ve denom.) $-b^2=-9$, $\therefore\ b^2=9$.

Now $b^2=a^2(e^2-1)$, $\therefore\ 9=4(e^2-1)$.]

Pages 231-232. Art. 141.

1. $4, 2\ ;\ (\pm\sqrt{5}, 0)\ ;\ \frac{\sqrt{5}}{2}\ ;\ 1\ ;\ \sqrt{5}x\pm4=0$. **2.** $\sqrt{2}\ ;\ 4\sqrt{2}\ ;\ 4$.

Page 232. Art. 142.

Ex. Tangent : $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$; Normal : $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

Pages 232-233. Art. 142.

2. $\frac{2ab}{a^2m^2-b^2}\sqrt{c^2-a^2m^2+b^2}.\sqrt{1+m^2}$; $c=\pm\sqrt{a^2m^2-b^2}$.

3. (*a*) $(2, -\frac{1}{2}), \left(-4, -\frac{7}{2}\right)$. (*b*) $3\sqrt{5}$.

4. $\left(5, -\frac{4}{3}\right)$. **5.** $c=\pm\sqrt{a^2m^2-b^2}$. **6.** $4x-2y\pm\sqrt{65}=0$.

7. (*i*) $2x-y\pm1=0$. (*ii*) $3x-y\pm\sqrt{6}=0$.

Pages 235-237.

1. (*i*) If P=0, the pt. (x', y') lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

(*ii*) If P>0 (*i.e.*, +ve), the pt. (x', y') lies inside the hyperbola.
(*iii*) If P<0 (*i.e.*, −ve), the pt. (x', y') lies outside the hyperbola.

3. $x-y+1=0,\ 3x-y+5=0.$ **4.** $3x-y+2=0,\ 3x+y-2=0.$

5. $x^2+y^2=a^2-b^2.$ **6.** $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$

7. (*a*) [See Art. 78.] (*b*) $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$

8. $\left(-\frac{A}{C}a^2, \frac{B}{C}b^2\right).$ **9.** $\left(3, \frac{8}{3}\right).$ **10.** $\left(-\frac{A}{C}a^2, \frac{4B}{C}a^2\right).$

Pages 238-239. Art. 144.

2. $\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1+\theta_2}{2}$; $\frac{x}{a}\sec\theta_1-\frac{y}{b}\tan\theta_1=1.$

3. $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2.$

Pages 239-241. Art. 145.

1. [**Hint.** Proceed as in Ex. 2, Art. 137, (*a*).]
2. [**Hint.** (*a*) Proceed as in Art. 137, (*b*).
(*b*) Proceed as in Art. 137, (*c*).]
3. [**Hint.** Proceed as in Art. 137, (*d*).]
4. [**Hint.** Proceed as in Art. 137, (*e*) and Cor.]
5. [**Hint.** Proceed as in solved Ex., Art. 137, (*e*).]
6. [**Hint.** Proceed as in Art. 137, (*f*).]
7. [**Hint.** Proceed as in Art. 137, (*g*).] **8.** $2x-3y=0.$
10. $125x-48y=481.$ [**Hint.** Proceed as in solved Ex. 2, Art. 84.]

MISCELLANEOUS EXAMPLFS ON CHAPTER VIII

Pages 241-243.

3. $3x^2-y^2-3a^2=0$, where $2a$ is the transverse axis.
[**Hint.** Let the transverse axis$=2a$. CA$=\frac{1}{2}$CS, $\therefore\ a=\frac{1}{2}ae,\ \therefore\ e=2.$]

4. $7y^2-24xy-24ax+6ay+15a^2=0$; $\left(-\frac{a}{3}, -a\right)$; $12x+9y+29a=0.$

5. (*a*) $4, 2\sqrt{2}$; $\sqrt{\frac{3}{2}}.$

(*b*) Transverse axis : $y-2=0$, conjugate axis : $x+2=0.$
[**Hint.** Proceed as in Misc. Ex. 4, Chap. VII.]

6. $(-1, 1)$; $\frac{5}{4}$; $(4, 1), (-6, 1)$; $5x-11=0,\ 5x+21=0.$

7. $(-1, -2)$; $2\sqrt{3}.$ **8.** $(1, 1).$

10. $x-y\pm\sqrt{a^2-b^2}=0,\ x+y\pm\sqrt{a^2-b^2}=0.$

11. $2x-y+1=0,\ 2x+y+1=0.$
[**Hint.** *Reject the imaginary values of m.*]

21. $32x-9y=0.$ [**Hint.** Use Art. 145, Ex. 12, Cor. 2.]

PUNJAB UNIVERSITY PAPERS

Pages 245-246. 1953. 1. (*a*) $\frac{af+bg+c}{\sqrt{a^2+b^2}}$. (*c*) $\left(\frac{78}{41}, \frac{128}{41}\right)$.

2. (*a*) $hx+ky=h^2+k^2$. **[Hint. Proceed as in Ex. 1, Art. 84.]**

(*b*) $\cos^{-1}\frac{5}{6}$. [See Misc. Ex. 37, Chap. V.]

3. (*a*) $yy_1=2a(x+x_1)$. [See Art. 111.] (*b*) **[Hint. The equation of the directrix is $x=-a$, $\therefore$ any pt. on the directrix is $(-a, y_1)$.]**

4. (*a*) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. [See Art. 136, Ex. 3, (*ii*).]

(*b*) [See Art. 136, Ex. 4.]

5. (*a*) [See Art. 138.] (*b*) [See Art. 139.] (*c*) $\frac{5}{4}$, $(\pm 5, 0)$, $\frac{9}{2}$.

Page 246. 1954. 1. (*a*) $\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]$, where (x_1, y_1), (x_2, y_2), (x_3, y_3) are the vertices of the triangle. [See Art. 9.]

(*b*) $x(y_1-y_2)-y(x_1-x_2)+(x_1y_2-x_2y_1)=0$. **[Hint. The area of the triangle whose vertices are (x, y), (x_1, y_1), (x_2, y_2) is zero.]** (*c*) 26.

2. (*a*) [See Art. 135.] (*b*) $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

3. (*b*) $\left(-\frac{g}{a}, -\frac{f}{a}\right)$, $\frac{\sqrt{g^2+f^2-ac}}{a}$. (*c*) $\left(\frac{3}{7}, -\frac{1}{7}\right)$. (*d*) $\frac{\sqrt{-95}}{7}$.

4. (*a*) $\left(-\frac{5}{2}, \frac{7}{2}\right)$. (*b*) [See Art. 117, (*d*).]

(*c*) [See Art. 117, (*e*), Method I.]

5. (*a*) $x-y+1=0$, $3x-y+5=0$. (*b*) $45x^2-36y^2-80=0$.

Page 247. 1955.

1. (*a*) $x'\cos\alpha+y'\sin\alpha-p$. [See Art. 44-I.] (*b*) $\left(\frac{5}{2}, \frac{5}{2}\right)$; $4\sqrt{2}$.

2. (*a*) (2, 3) ; 5. (*b*) $2x^2+2y^2-8x-10y-27=0$.

3. (*a*) $yy_1=2a(x+x_1)$. [See Art. 110.] (*b*) $\left(\frac{1}{3}, \frac{4}{3}\right)$.

4. (*a*) (*i*) $\frac{a^2x}{x'}-\frac{b^2y}{y'}=a^2-b^2$. [See Art. 123.] (*ii*) [See Art. 137,(*d*).]

(*b*) (5, 2), (−5, −2).

5. (*a*) (*i*) [See Art. 139.] (*ii*) [See Art. 145.] (*b*) $\sqrt{\frac{5}{3}}$.

Pages 247-248. 1956.

1. (*a*) (*i*) [See Art. 24.] ; (*ii*) [See Art. 43.] (*b*) $8x-y-1=0$.

2. $(-2, 3)$; $2\sqrt{3}$; $x^2+y^2+4x-6y+1=0$.

3. (*a*) (*ii*) $tx+y-2at-at^3=0$. (*b*) $\frac{80}{9}$.

4. (*a*) [See Art. 137, (*g*).] (*b*) (*i*) $4x-y\pm7=0$. (*ii*) $4x+y\pm7=0$.

5. (*a*) (*ii*) $x^2+y^2=a^2-b^2$. (*b*) $7y^2-24xy-24ax+6ay+15a^2=0$.

Page 248. 1957. 1. (*a*) [See Art. 30.] ; (x', y') is the given pt. thro' which the line is drawn, θ the angle which the line makes with the x-axis, and r the distance of the pt. (x, y) from (x', y').

(*b*) $5x+7y+11=0$. (*c*) $\sqrt{3}\,x+y-(\sqrt{3}-1)=0$, $y+1=0$.

2. (*a*) $a^2l^2+a^2m^2=n^2$. (*b*) $2x^2+2y^2-2x+8y+3=0$.

3. (*a*) (*i*) $\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}$. [**Hint**. See Art. 137,(*g*), Ex. 3,(*a*)]

(*ii*) [**Hint.** See Art. 137, (*g*), **Ex.** 3, (*b*).] (*b*) $\frac{\sqrt{5}}{3}$, $\left(\pm\frac{\sqrt{5}}{6}, 0\right)$, $\frac{4}{9}$.

4. (*a*) $t_1t_2=-1$ [See Misc. Ex. 31, Chap. VI.] ; [See Misc. **Ex**. 33, Chap. VI.] (*b*) $x^2-y^2+6ax+a^2=0$.

5. (*a*) [****Short-cut**. Let (x_1, y_1) be the pole of any tangent of the circle $x^2+y^2=a^2$ ...(1) w.r.t. the hyperbola $x^2-y^2=a^2$. The equation of the polar of (x_1, y_1) w.r.t. the hyperbola is $xx_1-yy_1=a^2$. $\because$ it touches the circle (1), $\therefore$? Find the locus of (x_1, y_1).]

(*b*) [**Hint.** The equations of the lines are $x-y=0$ ...(1), $x+y=0$ ...(2). The equation of any tangent to the hyperbola, in the m-form, is $y=mx+a\sqrt{m^2-1}$ ...(3). Find the area of the triangle formed by the lines (1), (2), (3). It will be found that this area $=a^2$.]

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